TALLINNA ÜLIKOOL

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Loodusteaduste osakond

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# STATISTICAL ANALYSIS OF THE STOCHASTIC SUPER-EXPONENTIAL GROWTH MODEL

# STOHHASTILISE SUPEREKSPONENTSIAALSE KASVUMUDELI STATISTILINE ANALÜÜS

Magistritöö

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Autor: .............................. „.......” 2011.õ.a.

Juhendaja: ........................ „.......” 2011.õ.a.

Lubatud kaitmisele: ........... „.......” 2011.õ.a.

Õppekava juht Tõnu Laas ...........................

Tallinn 2011

# Contents

# LIST OF ABBREVIATIONS AND SYMBOLS

| | |
|---|---|
| $\alpha$ | technological parameter |
| $r$ | growth rate of a population |
| $K$ | carrying capacity |
| $D$ | noise intensity |
| $\xi(t)$ | Gaussian white noise |
| $X(t)$ | population size (population abundance) |
| $P(x,t\|x_0,0)$ | conditional probability density function of the population size |
| $\langle X(t)\rangle$ | mean value of the population size |
| $\sigma(X(t))$ | standard deviation of the population size |
| $\delta(X(t))$ | relative variance (dispersion) of the population size |
| $t_c$ | critical time of the phase transition of the deterministic model |
| $t_p$ | population's prospective lifetime according to the stochastic model |
| $\langle T\rangle$ | the mean lifetime of the population according to the stochastic model |
| $P_T$ | conditional probability density function of lifetime of the population |
| HDI | human development index |
| CPDF | conditional probability density function |
| RFR | replacement fertility rate |
| TD | demographic transition |
| TFR | total fertility rate |

# INTRODUCTION

The focus of interest in the current thesis is to propose a stochastic model to describe the dynamics of human population in a perturbed environment with an increasing carrying capacity. The model in concern is aimed at being tractable and applicable for describing human population growth for the past 10 000 years. There is an important advantage of the long-term modelling because we can avoid complications of cyclic and chaotic dynamics that may apply to populations with short generations and high growth rates (Nilsson, 2006). The main objective in using stochastic methods for analyzing population dynamics is carried by the author's belief that this approach is more fundamental and more appropriate in biological modelling than modelling using only deterministic models.

The author aims to put forward an analytical model to describe the dynamics of human population change. Our approach is to treat the biological system of human population and its environment as a physical state in which there is an interaction between energy and matter. This type of physical state is highly organized and complex. The aim is to estimate the basic factors that have relevant effect on shaping the dynamics of human population and filter out the factors of little consequence to the overall behaviour. It is worth noting that the data analysis is not the primary aim of this research. There is significant amount of statistical research done with aggregated and disaggregated data by UN and various other authors.

We use the information gained from earlier research on human population dynamics done by previous authors and construct a phenomenological model to estimate future tendencies. This type of approach is ensued by the fact that the biological system has been treated as a complex physical state and by studying its physical properties we gain more insight in the biological system as a whole.

Analytical modelling or insight derived from system analysis, combined with simulation, often provides the most informative approach for understanding and interpreting population data. Statistical theory has established that the best predictive power is achieved by relatively simple models with a small or intermediate number of parameters, rather than by complex models, because limited data do not permit accurate estimation of many parameters. Ideal tool for describing human population dynamics would be a model, simple enough to be intelligible and complicated enough to be potentially realistic and empirically testable to be credible. Mathematical models are very useful for formalizing our thinking and to drawing the conclusions from proposed assumptions. To paraphrase Haldane "...an ounce of analysis is worth a ton of verbal argument" (Smith, 1965). It should also be kept in mind that the purpose of the current thesis is to explore the general behaviour of the population dynamics from a theoretical point of view. Hence, the parameter values of the figures presented in Chapter 3 and 4 have an illustrative value only.

For more than two centuries, scholars have devised mathematical methods for peering into our demographic future. Modelling human population dynamics is one of the oldest branches in ecology. The pioneering work of Thomas Robert Malthus in 1798 shaped the early views on human population growth. Malthus argued that any population, if left unrestricted, would grow geometrically (aka exponentially). Therefore, human population would continue to grow until it would become too large to be supported by the food grown on available agricultural land. Malthusian model was amended by Verhulst in 1838 by additional limiting term to mark the environmental constraints acting on the growth rate. There were oppositions to Malthusian paradigm from the eighteenth century onward. Marquis de Condorcet, Ester Boserupt, Julian Simon, and Marie-Jean-Antoine-Nicolas Caritat and many others argued that population growth can stimulate technological innovation. Nowadays, there are ample of *system dynamics* models for predicting the future of mankind, especially with the recent rise of computers and system analysis. However, predicting the dynamics of human population growth has proven to be an extremely difficult task. All of the previous methods have failed to predict events more than a decade or two in the future with quantitative accuracy (Cohen, 1995a).

Over the last 2 millenia, the annual rate of increase of global population grew about 50-fold, from an average 0.04% per year between A.D. 1 to its all-time peak of 2.1% per year around 1965 to 1970 (Cohen 1995b). The growth rate has since declined to about 1.09% in 2011 (CIA, 2011). Human influence on the planet has increased even faster than the human population growth. For example, while human population more than quadrupled from 1860 to 1991, human use of inanimate energy multiplied nearly 100-fold during the same time period (Cohen 1995b). Rapid population growth gave rise to notions such as *population explosion* and associated fear of socio-economic collapses, and ecological catastrophes.

To understand both the enormity and the subtlety of the consequences of rapid population growth that humanity faces today, it is helpful to have the variety of temporal and theoretical perspectives provided by archeology, demography, formal modeling, and interdisciplinary research—anthropological, political and economic, and mathematical. Demography proper has tended to lose itself within a disaggregated level of analysis, in hopes that understanding behavior at the individual level will provide the understanding needed for global problems. Contrarily, Johansen and Sornette (2001) argued that in order to get a meaningful prediction at an aggregate level, it is often more relevant to study aggregate variables than local variables that can miss the whole picture in favor of special idiosyncrasies. Thus, human population dynamics can be seen as a bridge between natural sciences and the humanities. This was most notably recognized by J. W. S. Pringle at Oxford, who was instrumental in establishing there the field of Human Sciences, essentially based on the analysis of human population structures from all perspectives (Mcbeth, Collinson, 2002).

Considering the role of environmental noise has become an integral part in modeling population dynamics. The theory of stochastic processes arose from the quantitative study of Brownian motion. Today it provides a huge arsenal of methods suitable for analyzing the influence of noise on wide range of systems. Most of the pioneering theoretical work was done by handful of physicists and mathematicians, such as Einstein, Smoluchowsky, Langevin, Wiener, Stratonovich, etc. Hence most of the early research centered around mathematical and physical questions. However, the last decade has seen enormous growth of applications to other sciences, especially chemistry, biology and social sciences. One of the main reasons for this *stochastics boom* may be the realization that the noise plays a constructive rather than the expected deteriorating role. These noise-induced, noise-supported or noise-enhanced effects can offer, at least to some extent, an explanation for insofar open problems. They may also pave the way to novel technological applications, such as noise-enhanced reaction rates, noise-induced transport and separation on the nanoscale, etc. (Goel, Richter-Dyn, 1974; Lande *et al.*, 2003; Freund, Pöschel, 2010)

The thesis is composed of four parts. In the first part we present the background to the topic of human population dynamics from demographic and ecological perspectives. The basic observation is that the human population has grown faster than exponentially for most of the known history. The concept of Earth's human carrying capacity and the problem of its estimation is introduced. The second part is an overview of various ecological models used to understand and simulate the dynamics of human population. The mathematical model for super-exponential growth, and its relevance to human population, is presented. We also demonstrate the limitations of the deterministic model. The third part concerns a stochastic super-exponential model. We present the assumptions behind the composition of the Langevin equation that describes the dynamics of a fluctuating population, solve the Fokker-Planck equation and analyze the behaviour of the conditional probability density function of the population size. Also, the mean value and the variance of the population size have been calculated. Similarly, we estimated the probability distribution function of the population's lifetime, as well as determined the population's mean lifetime and the prospective lifetime. In the fourth part we analyze the main results obtained from the third part and translate these results to human population dynamics. We also discuss various applications of the stochastic super-exponential growth model. The author emphasizes that using the theory of stochastic processes to analyze the temporal evolution of the population and its environment is the most significant part of this research.

The objectives of this M.Sc. thesis can be summarized as follows:

- Present the historical overview of the topic of human population dynamics and review the recent advances in the field.
- Demonstrate the adequacy of the super-exponential model for describing human population dynamics for the past 10 000 years and show how the results obtained from the mathematical model can be translated into the physical reality.
- Elaborate on the benefits of using a stochastic model for describing the evolutionary processes of real populations.
- Conduct a stochastic super-exponential model that takes into account the fluctuations present in the growth rate. The noise is introduced to the system multiplicative and is assumed to be Gaussian white noise a la Stratonovich.

- Analyze the main results obtained from the stochastic model and elaborate on the implications they have to the human population.

# 1. WORLD POPULATION GROWTH

## 1.1. Methodology for Historical Estimations of Human Population Growth

Historical knowledge of past human population sizes is a compendium of facts from various sources: archaeological remains, monuments, documents, observations of present human and nonhuman populations. Also, data from DNA sequence variation can be used to estimate the population size. Periods of population growth and decline leave signatures in the distribution of differences among individuals in a population. Population decline causes a loss of sequence diversity. Conversely, population growth causes the retention of sequences that otherwise would have been lost. Thus DNA sequence diversity may provide an instrument for examining prehistoric demography. To be more exact DNA sequence diversity provides us with the information of the *effective population size* and not the actual size of the population. The concept *effective population size* was introduced by the American geneticist Sewall Wright, who defined it as "the number of breeding individuals in an idealized population that would show the same amount of dispersion of allele frequencies under random genetic drift or the same amount of inbreeding as the population under consideration" (Wright, 1931). The underlying assumption here is that the effective population size is relevant to the actual population size. John Hawks, the Associate Professor of Anthropology at the University of Wisconsin—Madison alleged that the inbreeding effective size in ancient humans was likely somewhere between a third of the actual number of individuals, anywhere down to a tenth or less (Hawks, 2008).

Historical estimates of population numbers are often based on local, partial counts that served other purposes. In many early attempts, such as Han Dynasty, Ancient Egypt and in the Persian Empire the focus was on counting merely a subset of the people for purposes of taxation or military service. Until the late 18th century, only a few governments had ever performed an accurate census. Even today, there has not been complete census of the world's population, and demographers still rely on estimates to help construct world totals (Cohen, 1995a).

The fragmentary information based on archaeological excavations, early censuses, data from DNA sequencing etc., provide a flimsy basis for global population estimates. It has been stated that the Earth's population probably could not be known with an uncertainty of less than around 20% before the middle of the eighteen century (Coale, 1974).

## 1.2. An Historical Overview of Human Population Growth

The rate at which the human population grows has never been constant over the course of history. The population growth is influenced by a large amount of different factors and its temporal behaviour is stochastic and extremely complex. However, there are two essential features to describe human demographic growth in general:

- local and global growth differed formidably in the past, the correlation between different regions was especially faint in the past;
- relative and absolute population growth has varied significantly over time. Hypothetical models of constant relative growth or constant absolute growth are not sufficient to describe long-term human population dynamics.

The population biologist and ecologist Edward S. Deevey, Jr proposed that the human population experienced three periods of rapid rise (rapid by the past standard, not rapid compared to twentieth-century population growth):

- before 10,000 B.C. when people discovered how to use and make tools;
- from 8,000 B.C. to 4,000 B.C. when people developed agriculture and established cities;
- in the 18th century with scientific and industrial revolution.

Between these periods of rapid rise, Deevey (1960) proposed, that there were much longer periods of nearly stationary population size or very slow growth. However, Hammel and Howell (1987) have suggested that the search for general explanation of global population phenomena may be misguided, at least for the early stages of human history. Already from its earliest expansion, the population of *Homo Sapiens* was no longer a localized breeding population, but was rather a dispersed collection of such local populations.

The general smoothness of global population changes concealed some dramatic local fluctuations. As some local populations had the upward drift, others declined or became extinct. Huff et al (2010) showed that the effective population size of human ancestors living before 1.2 million years ago was around 18,500. Also, modern humans are survivors of dramatic reductions from 100,000 to 10,000 years ago (Rogers, Harpending, 1993; Harpending *et al.* 1993; Gibbons 1993).

It is not known when exactly did an ancestral group become close enough to present *Homo sapiens*. Hominids, in the sense of the lines of descent leading to today's great apes and humans, originated approximately 17 million years ago. The human lineage clearly existed 4 million years ago. Fossil tracks indicate that proto-humans were bipedal 3 million years ago. Flaked-stone tools manufactured by proto-humans date back 2.4 million years. Humans with modern anatomy existed perhaps 120,000 years ago. There were about 2 to 20 million people living at 10,000 B.C. and around 170 to 330 million people at A.D. 1. It is estimated that the average growth rate prior 10,000 B.C. was roughly 0.0016% per year, while from 10,000 B.C. to A.D. 1 the average growth rate was roughly 0.021% per year. At a minimum, the average rate of population growth increased by a factor of 13 after the local agricultural evolution, compared to the earlier period. (Schick, Toth, 1993)

On Fig. (1) historical estimates of global world population from 10 000 B.C to 1950 A.D. by different authors have been illustrated. On Table (1) we have illustrated the major milestones of the world population: years have been listed when another billion people have been added to the world. It is evident that most of the population growth has taken place in the past century.

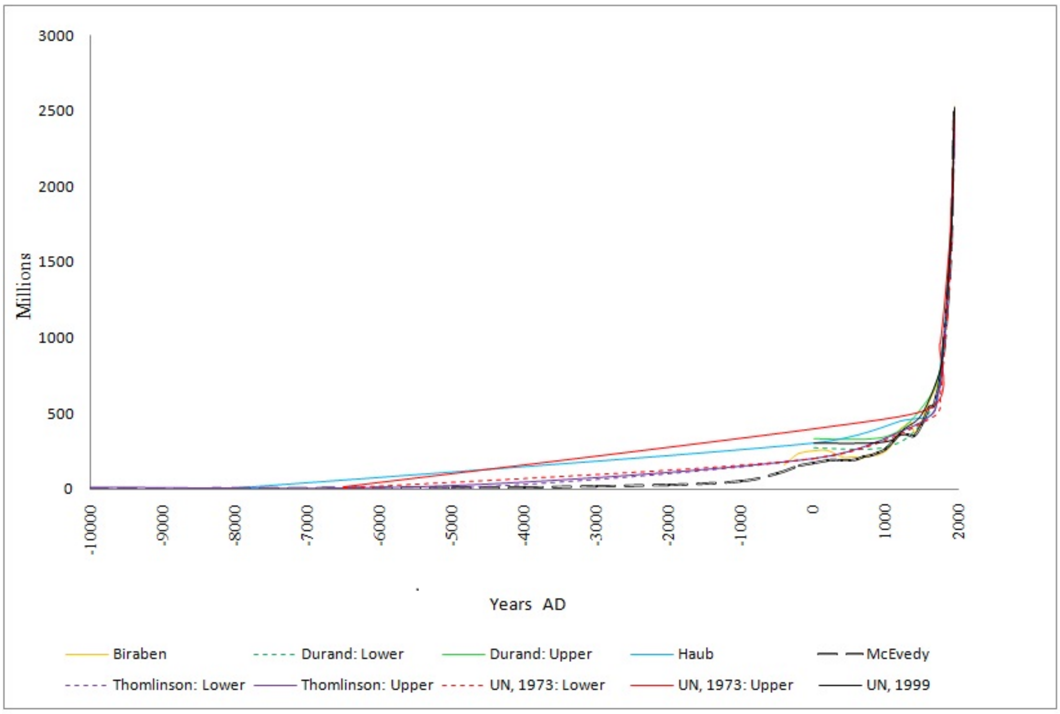

Figure 1: Historical Estimates of World Population: from 10 000 B.C. to 1950 A.D. Sources of the data: (Biraben, 1980), (Durand, 1974), (Haub, 1995), (McEvedy *et al.* 1978), (Thomlinson, 1975), (UN, 1973), (UN, 1999).

| World population (billions) | Year | Years elapsed |
|---|---|---|
| 1 | 1804 | |
| 2 | 1927 | 123 years |
| 3 | 1960 | 33 years |
| 4 | 1974 | 14 years |
| 5 | 1987 | 13 years |
| 6 | 1999 | 12 years |
| 7 | 2011 | 12 years |

Table 1: World Population milestones. Source of the data: (UN, 1999).

From A.D. 1 to 1650, global population almost doubled to 500-600 million. For 1750, human numbers already ranged from 650 to 850 million. For 1950 it had grown to 2.5 billion. The average growth rates are 0.039% per year from 1 A.D. to 1750 and 0.70% from 1750 to 1950. In the next quarter century, the average annual growth rate more than doubled, to 1.9% per year, as the population rose to 4.1 billion by 1975. Between 1975 and 1990, population rose to 5.3 billion, at an average growth rate of just under 1.8% per year. (Cohen, 1995a)

Historical estimates of different authors have been plotted in Fig. (1), (2), and (3). The Fig. (2) and (3) show logarithm of the estimated world population as a function of linear time, such that an exponential growth rate would be qualified by a linear increase. In contrast, these figures imply a strong upward curvature characterizing growth that is faster than exponential.

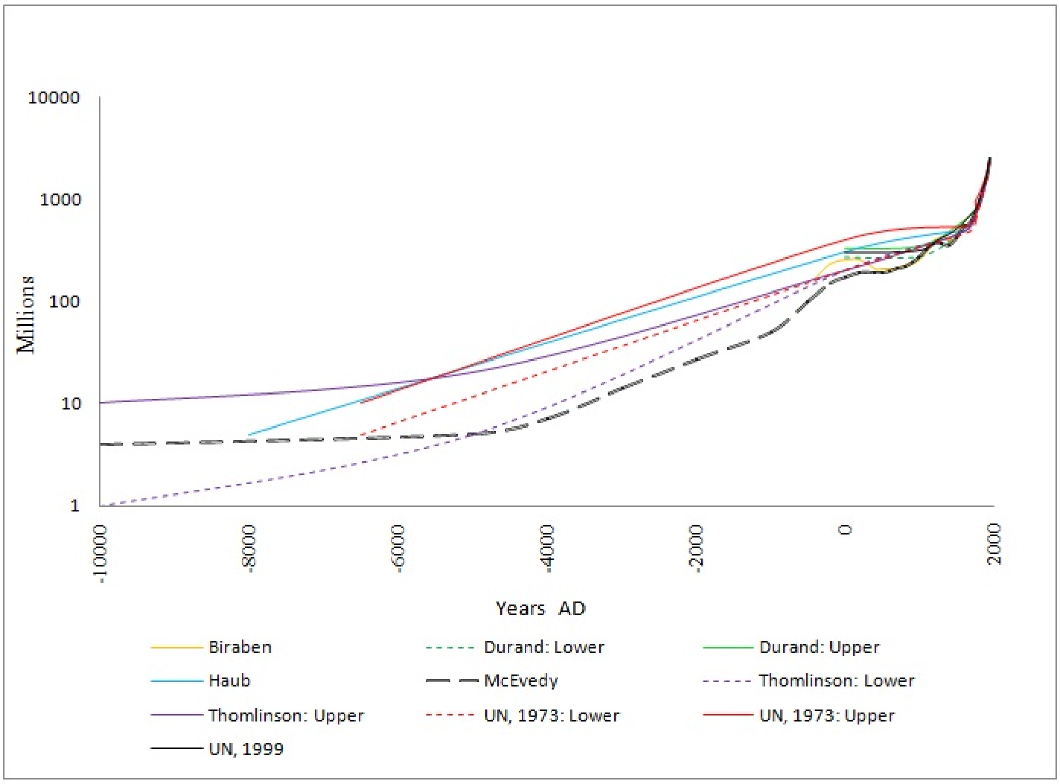

Figure 2: Historical Estimates of World Population: from 10 000 B.C. to 1950 A.D. Semi-logarithmic representation. Sources of the data: (Biraben, 1980), (Durand, 1974), (Haub, 1995), (McEvedy *et al.* 1978), (Thomlinson, 1975), (UN, 1973), (UN, 1999).

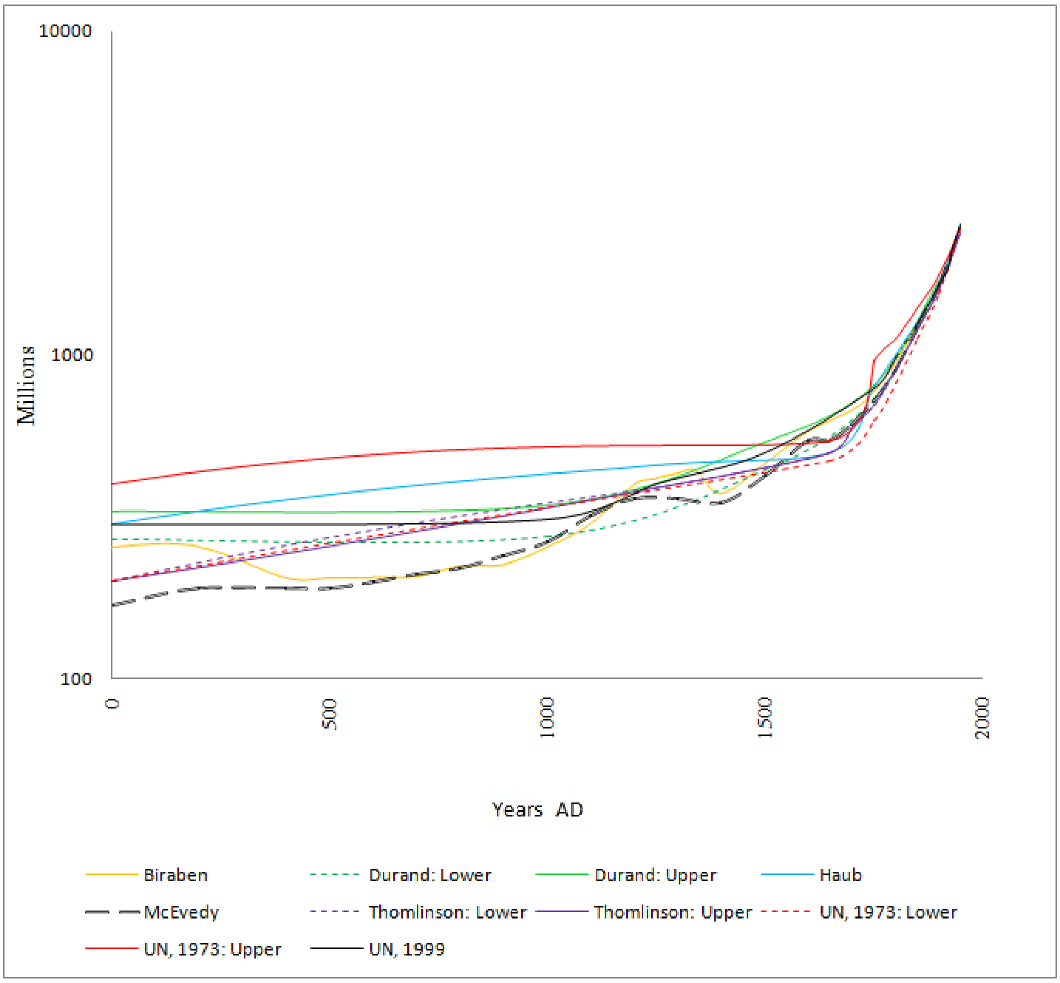

Figure 3: Historical Estimates of World Population: from 0 to 1950 A.D. Semi-logarithmic representation. Sources of the data: (Biraben, 1980), (Durand, 1974), (Haub, 1995), (McEvedy *et al.* 1978), (Thomlinson, 1975), (UN, 1973), (UN, 1999).

There are numerous biologically interesting consequences of the super-exponential population growth in humans. A recent DNA sequencing survey on nearly 15,000 humans revealed that rare variants are much more common than expected from standard population genetic theory. It has been showed via probability generating that the impact of such population growth is highly advantageous for the survival of a new mutant. The population growth greatly increases the probability of survival of a mutant, and highly deleterious mutants can have a finite probability of persisting indefinitely. This can help explain the anomaly of "missing heritability" in genetic epidemiology studies and change current understanding of genetic risk factors to disease in humans. (Coventry *et. al.*, 2010)

Now, in the 21st century, during the era of technological optimism, it is believed that humans have became extinction free (Steinmann *et al.*, 1998; Kogel and Prskawetz, 2001; Galor and Weil, 2000). Optimism that follows from an unveiling success of technological improvement has fuelled the idea that the human population can be freed from its Malthusian trap has biased the modern view of ecologist and futurists. However, from case studies, it can be seen that there is ground for pessimism.

Technological pessimists or *ecological realists* see practical limits to global and regional carrying capacity in agriculture, and maintain that the world is close to, or may even have passed, these limits. Harris and Kennedy have shown that the projections for world agriculture in the first half of the twenty-first century vary widely, largely depending on assumptions on yield growth. They investigated the pattern of yield growth for major cereal crops and present concluded that the supply and demand in the twenty-first century follows a logistic rather than an exponential or super-exponential model. These results imply the ecological limits on soil fertility, water availability, and nutrient uptake and the world is indeed close to carrying capacity in agriculture. Also, specific resource and ecological constraints are of particular importance at the regional level. There is evidence that the supply-side strategy of increased production has already led to serious problems of soil degradation and water overdraft, as well as other ecosystem stresses. The demand-side issues of population policy and efficiency in consumption are crucial to the development of a sustainable agricultural system. (Harris and Kennedy, 1999)

## 1.3. The Total Fertility Rate and Socio-Economic Development

In this section we will elaborate on the key issues of the world population distribution, dynamics of fertility rates on global and local scale, and shed light onto the relationship between the total fertility rate (TFR) and socio-economic development.

The TFR is defined as an average number of children that would be born to a woman over her lifetime if she were to experience the exact current age-specific fertility rates. There is another important demographic quantity - the replacement fertility rate (RFR). RFR denotes a fertility rate at which women have just enough babies to replace themselves and their partners. The replacement fertility rate is roughly 2.1 births per woman for most industrialized countries, but ranges from 2.5 to 3.3 in developing countries because of higher mortality rates.

The TFR is a better index of fertility than the Crude birth rate (annual number of births per thousand people) because it is independent of the age structure of the population. It may take several generations for a change in the total fertility rate to be reflected in birth rate, because the age distribution must reach equilibrium. For example, a population that has recently dropped below its replacement fertility rate will continue to grow, due to the high fertility produced by large numbers of young couples who would now be in their childbearing years. This phenomenon has its effect for several generations and is known as the population momentum or population-lag effect. The population dynamics for the past few decades in China is an illustrative example of the population momentum. The Chinese government implemented the one-child policy back in 1979, under which most of the country's couples were encouraged to have only one child. However, there were exceptions made, depending on geography or ethnic minority status, as for some women it was allowed to have more than 1 child. Nevertheless, on average, Chinese families had less than 2 children per woman. By 2005, the fertility rate of the mainland Chinese women had fallen to about 1.75. In spite of all these efforts, China's population still continues to increase even today, making out roughly 20% of the world total population, with over 1.3 billion people.

Most of the world population is located in Asia and Africa. Asia accounts for over 60% of the world total with more than 4 billion people. Africa follows with 15% of the world total, with 1 billion people. Europe makes up 11% of the world's population, with 733 million people. Latin America and the Caribbean region follow with 589 million (9%), Northern America is home to 352 million (5%) and Oceania to 35 million (less than 1%). The world population distribution by country have been illustrated on Fig. (4)

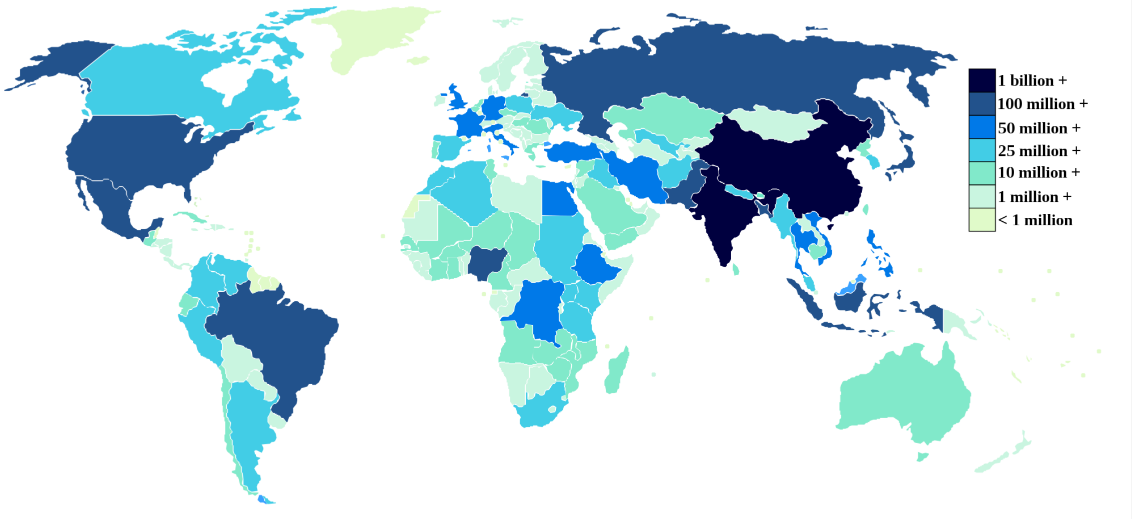

Figure 4: Population sizes by countries in the year 2010. Source: Wikipedia.

The inverse correlation between wealth and fertility within and between nations is a well known empirical fact. Nations or sub-populations with higher GDP per capita are observed to have fewer children, even though a richer population can support more children. On Fig. (5) the negative correlation TFR and GDP per capita have been illustrated. As a result of this connection between development and fertility decline, more than half of the global population now lives in regions with below-replacement fertility (Wilson, 2004).

This phenomena is also referred to as the demographic-economic paradox, because it contradicts the Malthusian paradigm of population growth - greater means would necessitate the production of more offspring. Indeed, most of population growth today happens in developing countries. The negative association of fertility with economic and social development has therefore become one of the most solidly established and generally accepted empirical regularities in the social sciences (Bryant, 2007; Lee, 2003; Bongaarts, Watkins, 1996). On Fig. (6) World's fertility rates by countries have been illustrated during the period 2005-2010.

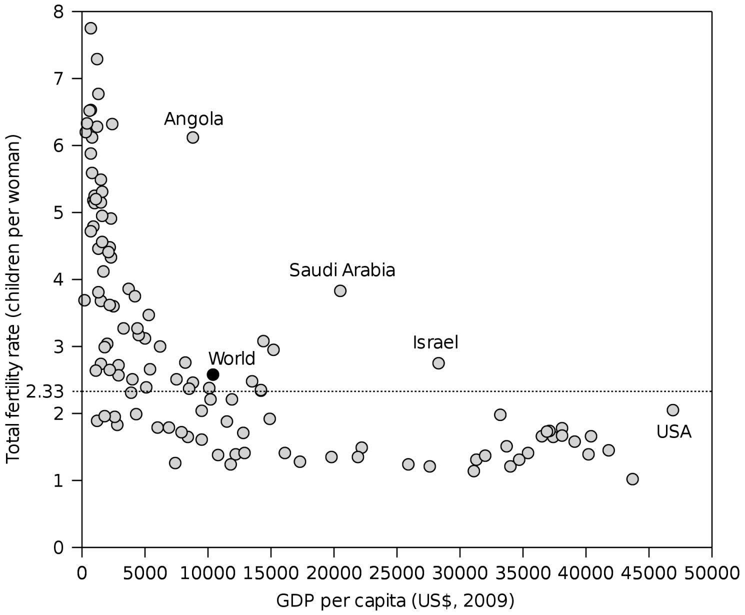

Figure 5: TFR vs. GDP per capita of the corresponding country, 2009. Only countries with over 5 Million population were plotted, to reduce outliers. Source: Wikipedia.

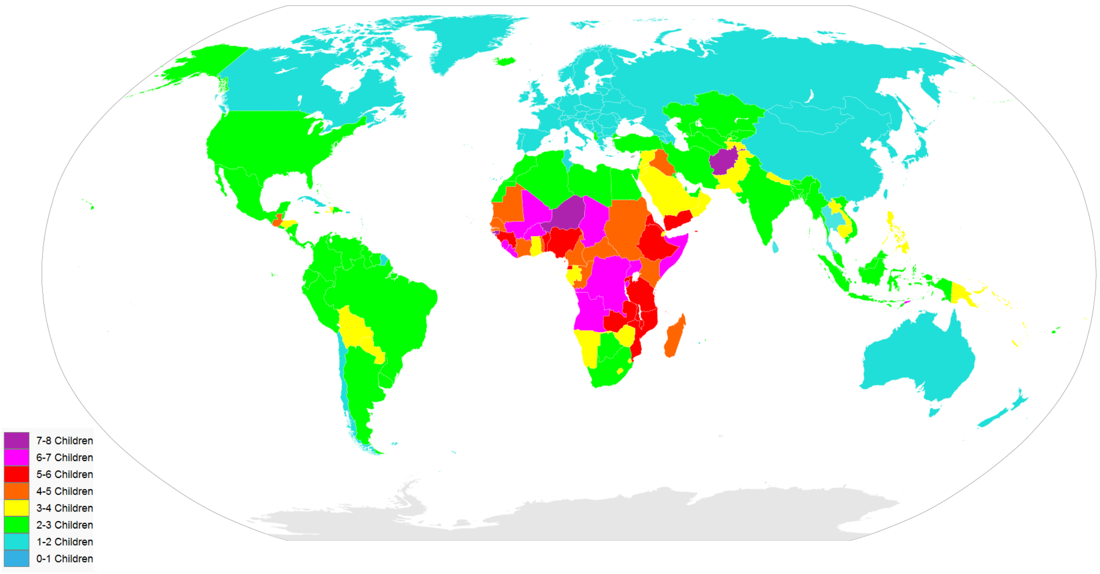

Figure 6: Fertility rates by countries during the period 2005-2010. Source: Wikipedia.

During the twentieth century, the global population has gone through prodigious increases in economic and social development that coincided with substantial declines in human fertility and population growth rates (Lutz *et al.*, 2004). The demographic transition (DT) is a model used to represent the transition from high birth and death rates to low birth and death rates as a country develops from a pre-industrial to an industrialized economic system. The theory is based on an interpretation of demographic history developed in 1929 by the American demographer Warren Thompson (1887–1973). Theory of DT suggests that there is a potential for demographic transition in countries that are currently poor if they became economically and socially better off. As a consequence, the global fertility rates would also decrease and the world population growth would eventually stop.

The theory of DT has been popularized in media with many well-known confident sentiments such as the "development is the best contraceptive" or "take care of the people and population will take care of itself". Yet popular adoption lags far behind the scientific rigour and the elaboration of the theory itself.

The theory of DT was originally developed based on an interpretation of demographic history developed in 1929 by the American demographer Warren Thompson (1887–1973). This nearly a century long theory has increasingly been put under scientific doubt, because of its weak explanationary power to the past events (Teitelbaum, 1975; Cohen, 1995a). Only modest claims are now made for transition theory as an explanation for the same demographic processes from which the theory was originally drawn - 19th-century Europe. The more detailed data about the European fertility decline that have been assembled in the last three decades show the immense complexity of these processes. Ultimately, demographers could not explain the fertility decline in Europe and in populations of European origin by geographic, social, economic or demographic characteristics, therefore they resorted to cultural explanations (Cohen, 1995a).

Whether it is appropriate to apply the theory of DT to modern Asia, Africa and Latin America is still an open discussion. There are many arguments that do not support the latter proposition. For example, since the decline in fertility rates in places where it has already occured is not understood, we cannot identify the precise conditions that lead to it, and therefore there are no grounds for asserting that today's less developed countries will obey these conditions. Furthermore, European fertility transition took place in setting of rich resources and opportunities for expansion and colonization. As well as, European countries, especially those of western Europe, never had to struggle with population growth rates as high as those of some poor countries recently. (Cohen, 1995a)

Maurice King (1993) argued that poor countries may be prevented from completing a DT by *demographic entrapment.* He proposed that if a country's mortality falls much more rapidly than its fertility, the population will expand so rapidly that the ecological supports of the country, such as topsoil, trees, biological diversity, will be consumed before a rise in wealth triggers a fertility decline. If the population cannot transcend to more favorable conditions, the country may be trapped with high fertility, a rapidly growing population and scarce ecological resources. He also suggested that it would be dangerous to lower infant and childhood mortality in countries with huge fertility rates, because it would only accelerate the population growth and deepen the trap.

Yet most of the influential demographic projections suggest that the global human population will stabilize at about 9–10 billion people by mid-century (UN, 1999; Lutz *et al.,* 2004). These projections rest on two fundamental assumptions (DeLong *et al.*, 2010):

- The energy needed to fuel development and the associated decline in fertility will keep pace with energy demand far into the future. This assumption relies on the theoretical and empirical relation between energy-use and population growth rates.

- Demographic transition is irreversible such that once countries start down the path to lower fertility they cannot reverse to higher fertility.

DeLong et al (2010) have criticized both of these wide-spread assumptions. They showed that the current projected future energy supply rates are far below the supply needed to support a global demographic transition to zero growth, suggesting that the predicted leveling-off of the global population by year 2050 is unlikely to occur, in the absence of a transition to an alternative energy source. It was also proposed that the population may actually fall to a lower level than was sustainable before the overshoot. The reason for unsustainable consumption while in overshoot allowed the species to use more non-renewable resources and to further poison their environment with excessive wastes. It is a common understanding of ecology that overshoot degrades the carrying capacity of the environment.

The inverse correlation between economic and social development has also been recently put under question. Myrskylä et al (2009) have showed that the relationship between the total fertility rate (TFR) and socio-economic development, which is measured by the human development index (HDI)[1], resembles a J-shaped curve. The cross-country association between total fertility rate (TFR) and HDI in 1975 and 2005 is shown in Fig. (7). Although development continues to promote fertility decline at low and medium HDI levels, they showed that at advanced HDI levels, further development can reverse the declining trend in fertility.

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[1] The HDI is the primary index used by the United Nations Development Programme (UNDP) to monitor and evaluate broadly defined human development, combining with equal weight indicators of a country's health conditions, living standard and human capital.

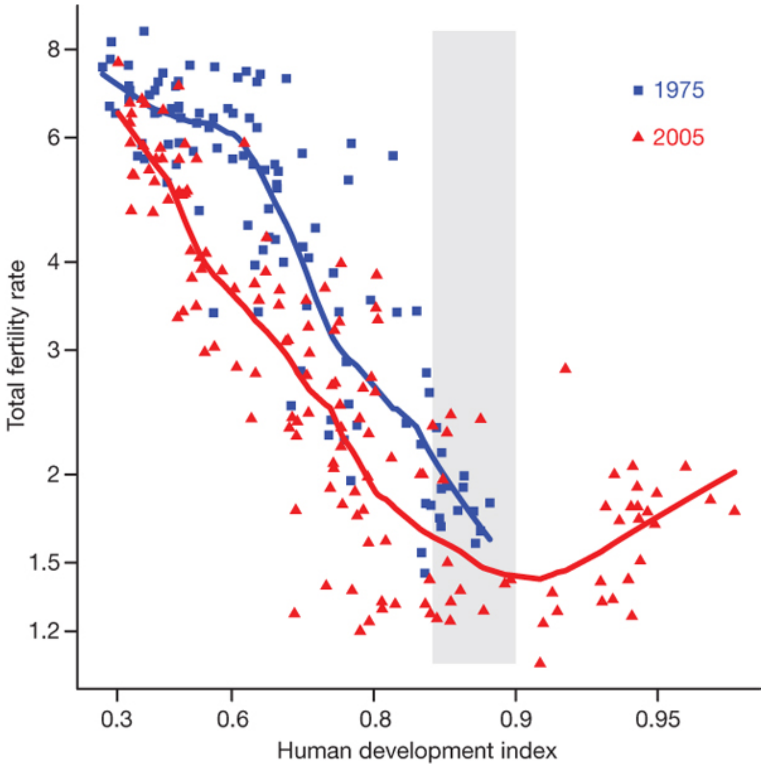

Figure 7: Cross-sectional relationship between TFR and HDI in 1975 and 2005. (Myrskylä *et al.*, 2009).

## 1.4. Earth's Human Carrying Capacity

Since the invention of fire the human species has sought to improve and enrich the environment and the resources it provides through its own intelligence. Hunting and gathering could certainly not have allowed the survival of many more than several million people. The European agricultural system could only support a little over a hundred million people who lived on the continent pior the Industrial Revolution. However, the correlation between technological improvement and population growth may not be as obvious due to two fundamental reasons. Firstly, because of the intermittent unpredictable events, such as natural disasters and epidemics and secondly, demographic mechanisms that determine the population growth change slowly over time and does not adapt as easily to rapidly evolving environmental conditions. (Livi-Bacci, 2006)

There are ample of different estimates of how many people the Earth could support. Cohen (1995a) put forth an exhaustive compendium of 65 estimates by different authors in chronological order. These estimates are plotted as a function of time on Fig. (8). Remarkably, there is no clear increasing or decreasing trend in the estimated upper bounds and the scatter among the estimates seems to increase with the passage of time, as more and more extreme. The increasing scatter is the opposite of the progressive refinement and convergence that would ideally occur as time passes when some natural quantity is being estimates. The increase of variation implies that the Earth's human carrying capacity is not a constant of nature. (Cohen, 1995a)

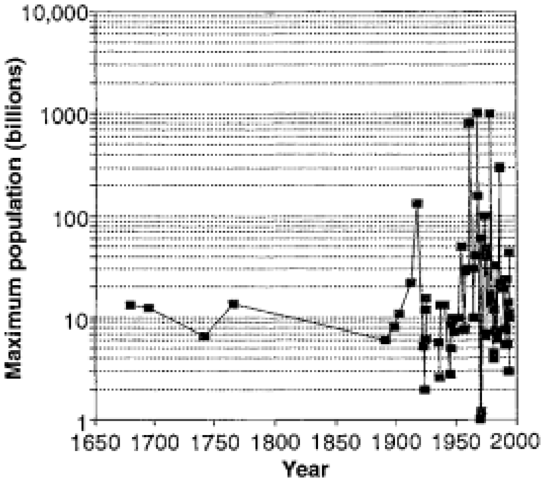

Figure 8: Estimates of how many people the Earth can support, by the date at which the estimate was made. When an author gave a range of estimates or indicated only an upper bound, the highest number stated is plotted here. J. H. Fremlin's (1964) highest estimate of $10^{18}$ is omitted, because it would be off the scale. (Cohen, 1995b)

The estimates of how many people the Earth can support vary from less than one billion to more than 1000 billion. The low estimates indicate that more people are already on the Earth that can be sustained. The diversity of the estimates suggests that the authors who made them were trying to measure different concepts, or had different data at their disposal or made different assumptions when data was incomplete. Cohen (1995b) presented six basic quantitative methods of estimating human carrying capacity:

1. Dividing Earth's land into regions and calculating a maximum supportable population density for each region separately. Summing over all regions yields a maximum supportable population of Earth.

2. Extrapolating from the historical data to make predictions for the future. The causal factors responsible for the changes in birth rates and death rates are however, still not well understood.

3. Focusing on a certain factor that constrains the population numbers, such as the available food supply. Besides food, other factors proposed as sole constraints on human population include energy, biologically accessible nitrogen, phosphorus, fresh water, light, soil, space, diseases, waste disposal, nonfuel minerals forests, biological diversity, and climate change. Such single-factor based analysis's are however very limited, because they don't take into account other factors that may come into play.

4. Several authors reduced multiple requirements to the amount of some single factor. For example, in 1978 Eyre reduced requirements for food, paper, timber, and other forest products to the area of land required to grow them.

5. Population growth is constrained by multiple independent factors and the constraint on the growth rate is determined by Liebig's law of minimum. The law asserts that under steady state conditions, the population size of a species is constrained by whatever resource is in shortest supply. However, Liebig's law is useless when limiting factors fluctuate, because different factors may be constraining at different times.
6. Several authors have treated population size as constrained by multiple interdependent factors. This approach sets fourth large sets of differential equations (deterministic or stochastic), which are usually solved numerically. System models of human population size and its carrying capacity have often embodied relations and assumptions that are neither mechanistically derived nor quantitatively measured.

Johansen and Sornette (2001) showed that the world economic output, which is strongly correlated with Earth's carrying capacity have also been growing faster than exponentially for most of the known history. The semi-logarithmic plot on Fig. (9) illustrates the faster than exponential growth in the estimated gross domestic product (GDP) of the World. Over a shorter time period, a super-exponential growth is also observed for a number of economic indicators such as the Dow Jones Average since the establishment of the USA in 1790, S&P since 1871, as well as for a number of regional and global indexes since 1920, including the Latin American index, the European index, the EAFE index and the World index (Johansen, Sornette, 2001).

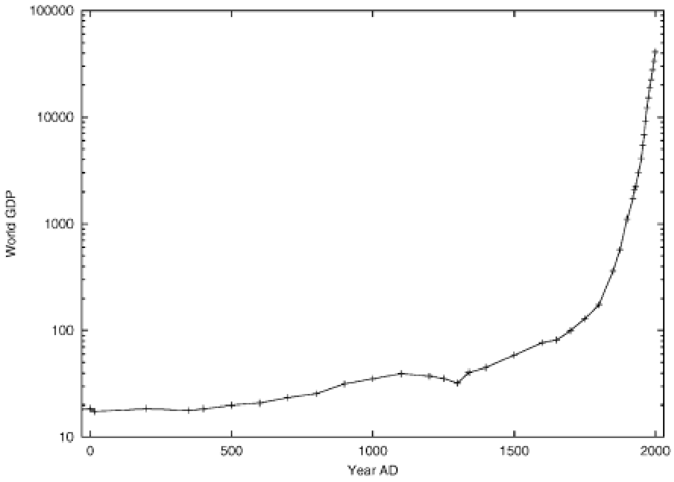

Figure 9: Semi-logarithmic plot of World GDP from year 1 A. D. until 2000. (Johansen, Sornette, 2001)

Another issue is that the *human carrying capacity of the world* is not an unitary concept on which there is universal agreement. The carrying capacity of nonhuman species has been defined in at least nine different ways, none of which are adequate for humans. One of the most widely used definitions of carrying capacity is "number of individuals in a population that the resources of a habitat can support; the asymptote, or plateau, of the logistic and other sigmoid equations for population growth" (Ricklefs, 1990). Earth's capacity to support people is determined by wide range of factors that range from economics and environment to culture and demography. Many of the natural constraints are not fully understood and are dependent upon human collective and individual choices, such as the level of material well-being, consumption level, family structure etc. Human carrying capacity is therefore a dynamic and uncertain quantity, and its real measure is being created by our choices today and in the future. Human decisions cannot be captured by ecological notions of carrying capacity that is suitable for nonhuman populations.

Cohen (1995a) has taken notice of the fact that recent demographers have eschewed questions of human carrying capacity. He proposed that the present theoretical framework of demography focuses attention on the composition and growth of populations and diverts attention from their absolute size. The theoretical framework of economics focuses attention on problems of allocation and distribution and away from problems of the scale of economy. An enlightening theory of demographic and economic scale would be helpful to understand the problem of sustainability of human population.

There are always a lot of assumptions made when demographic projections are being made. However, many of the causal relationships between various agents relevant in demographic projections are not correctly understood.

Cohen (1995b) has bestowed consideration upon the need for developing a stochastic approach to measuring Earth's human carrying capacity. Although a probabilistic measure of human carrying capacity has been developed for local populations in the Amazon, no probabilistic approach to global human population carrying capacity has been developed (Fearnside, 1986). However, incorporating stochasticity can strongly affect our current understanding of sustainability

A major challenge in the coming decades is to understand how demographic, economic and cultural changes will interact with Earth's physical, chemical and biological environments.

# 2. A BRIEF HISTORICAL OVERVIEW OF POPULATION MODELS FOR WORLD POPULATION

## 2.1 Early Models for Human Population Growth

The purpose of this chapter is to provide a brief historical background to main notions and models of human population dynamics.

Taxation and military prognoses were most doubtingly one of the main tasks of early mathematicians. However, there is not much known about the demographic studies and models from early civilizations (Nilsson, 2006). In 1767, Leonhard Euler produced the first mathematical model to describe human population growth. This was closely followed, in 1772 by Johann-Heinrich Lambert, who gave a mathematical formulation of the law of mortality. Thomas Robert Malthus proposed what has later been called the Malthusian law in 1798 where population growth is exponential. Malthus argued that growth of population is geometrical whereas growth of resources is at most linear which will cause famine. He concluded that war, disease, and famine are inevitable. The differential equation for population size $x = x(t)$ at the time $t$ is

$$\frac{dx}{dt} = r \cdot x, \tag{1}$$

where $r$ is growth rate. The solution to Eq. (1) is

$$x = x_0 e^{r(t-t_0)}, \tag{2}$$

where $x_0$ is the initial population size at time $t_0$. Malthus opposed the optimism of the French mathematician and philosopher Marquis de Condorcet (1743-1794), by whom the human mind paved the way of removing all obstacles to human progress. Malthus predicted that the population growth will always promptly win a race against the rate of growth of resources. He did not foresee how much people can expand the human carrying capacity of Earth, including but not limited to food production.

Pierre François Verhulst corrected the Malthusian unbounded growth model in 1838 by assuming a finite carrying capacity that exerts a damping force to the exponential growth. This model is called the logistic growth model and logistically growing population evolves according to

$$\frac{dx}{dt} = r \cdot x\left(1 - \frac{x}{K}\right), \tag{3}$$

where $K$ denotes the carrying capacity. The solution for the Eq. (3) is

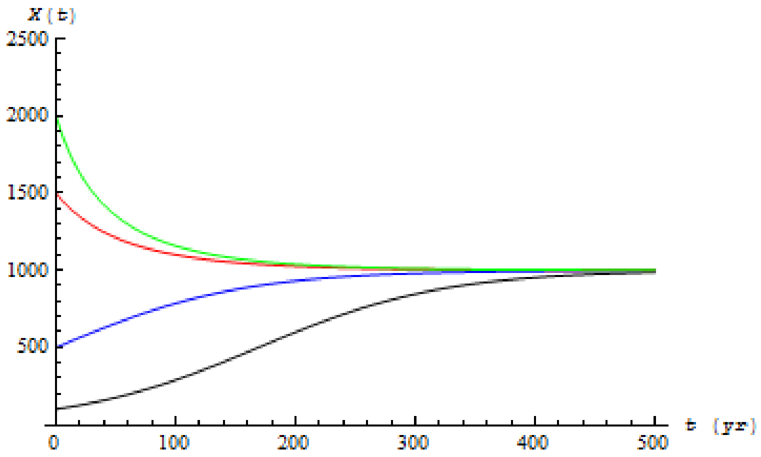

Figure 10: Logistic Growth model. The parameter values carrying capacity $K = 1000$, growth rate $r = 0.013$, initial time $t_0 = 0$. Black line: $x_0 = 100$; blue line $x_0 = 500$ ; red line: $x_0 = 1500$; green line: $x_0 = 2000$.

$$x = \frac{K}{1 + (K/x_0 - 1)e^{-r(t-t_0)}}. \tag{4}$$

The differential equation (3) for logistic growth was more or less forgotten for many decades and was rediscovered by Pearl and Reed in 1920. It has an attractor when $r > 0$ and $K > 0$. This means that all positive initial populations will eventually end up in $x = K$. Logistic growth at different initial conditions has been illustrated on Fig. (10).

Allee pointed out in 1931 that in many cases there is a minimum population size needed for positive growth, and that the population will decrease when below this size. This effect may be due to social interactions between individuals or they are more vulnerable to environmental fluctuations. The Alle effect creates growth rates that are negative in a range $(0, x_c)$, where $x_c$ is the minimum population size needed for positive growth.

In 1939 Alfred J. Lotka made important contributions to age-structured populations by using integral equations. Let $B(t)$ denote the number of births at time $t$, $b(a)da$ the expected reproductive output of an individual of age $[a, a+da]$, and $x(a,t)$ the number of individuals at age $a$ at time time $t$. Lotka Integral Equation yields

$$B(t) = \int_0^\infty x(a,t)b(a)da. \tag{5}$$

Assuming stationary age-structure implies that $x(a,t)$ is the number surviving individuals born at $t-a$, from the number of births $B(t-a)$. Introducing the probability to survive to an age $a$, $l(a)$. Since $n(a,t)$ are the surviving newborns at time $t-a$, therefore

$$B(t) = \int_0^\infty B(t-a)l(a)b(a)da. \tag{6}$$

## 2.2. Super-Exponential Growth Model Triggered by Technological Innovations

Cohen (1995b) introduced the idea to treat carrying capacity as a function of time $K(t)$ that depends on the number of people and how much they contribute to the increase or decrease of the carrying capacity. He considered a logistic growth model of the form

$$\frac{dx}{dt} = r \cdot x\left(K(t) - x\right). \tag{7}$$

He formulated a differential equation to describe the changes in carrying capacity and coined it as a Condorcet equation after French philosopher Marquis de Condorcet. Condorcet equation is defined as

$$\frac{dK(t)}{dt} = c\frac{dx}{dt}, \tag{8}$$

where $c$ is the Condorcet parameter. Condorcet parameter can take on different values:

- If $c > 1$, then it follows that each additional person increases the human carrying capacity on average enough to support itself plus a little extra. In that case logistic function (7) would grow super-exponentially, because the increase of the term $K(t) - x$ in time.
- If $c = 1$, then each additional person increases the human carrying capacity on average just enough to support itself. In that case logistic function (7) would grow exponentially, because the term $K(t) - x$ remains constant in time.
- If $c < 1$, then each additional person on average is not able increases the human carrying capacity enough to support itself. Therefore the function (7) will grow logistically and the population growth decelerated to zero.

The semi-logarithmic plots presented on figures (2) and (3) for the human population growth for the past 10,000 and 2,000 years cannot be approximated as exponential growth. The Fig. (2) and (3) imply that the growth has clearly been faster than exponential, i.e. super-exponential. Therefore the Condorcet parameter for humans is $c > 1$. Johansen and Sornette (2001) devised a mathematical framework to describe super-exponential growth of humans:

$$\frac{dx}{dt} = r[x(t)]^{\alpha+1}, \tag{9}$$

where $\alpha > 1$ is a parameter describing technological advancement, $t$ denotes time and $r$ is the growth rate (Malthusian parameter). The particular solution for Eq. 9, if the initial condition $x(0) = x_0$ holds, can be written as follows

$$x(t) = \frac{1}{\sqrt[\alpha]{x_0^{-\alpha} - \alpha r t}}. \tag{10}$$

The behaviour of the function given by Eq. 10 indicates that a population growing at a super-exponential mode will encounter a catastrophe within a finite time. Moreover, the afore-presented function has a singularity at the critical time

$$t_c = \frac{1}{x_0^{\alpha} \alpha r}. \tag{11}$$

The critical time $t_c$ verifies the moment of time when the demographic singularity takes place, i.e. $x(t_c) = \infty$. Johansen and Sornette (2001) estimated that the best fit for the demographic singularity is 2052 ± 10.

Although the catastrophic event is mathematically described as the singular state for the population, it should be noted that infinite population on finite Earth is evidently a mathematical idealization. In the present case, the demographic singularity should be interpreted as an abrupt regime shift to a qualitatively different state of existence. The transformation of the population will be more comprehensive than any other in the complete history of *Homo sapiens*. In the succeeding section, a more elaborative discussion of the interpretations of the demographic singularity will be presented.

The existence of finite-size and dissipation-like effects, which are common in these types of processes, will possibly smoothen out the abruptness of the transition. The all-time peak in the global population growth rate in 1965-1970 and then its initial decline can be interpreted as an entry to the transition era to a new regime. Interestingly, scholars failed to foresee the momentous event of decline of global growth rate prior and even failed to recognize it as it was happening (Cohen, 1995a).

The existence of finite time singularity is the direct result of the accelerating growth rate of the global world population for the past 10 000 years. Bettencourt et al (2007) showed using an extensive amounts of empirical evidence that there is a strong correlation between urban population growth and its technological innovation rate. Social organization and dynamics relating urbanization to economic development and knowledge creation, among other social activities, are very general and appear as nontrivial quantitative regularities common to all cities, across urban systems. Population growth driven by innovation implies, in principle, no limit to the size of a city and these urban populations grow super-exponentially.

The accelerating character of the growth rate describing human population can be seen as the inherent property also present in other evolutionary processes. For example Markov and Korotayev (2008) showed that growth of the Phanerozoic marine biodiversity at genus, family, and species levels, as well as Phanerozoic continental biota at genus and family levels, is in accord with hyperbolic growth. Based on the analogy with macro-sociological model for human population growth, the hyperbolic character of biodiversity growth can be similarly accounted for by a non-linear second-order positive feedback between the diversity growth and community structure complexity. There are two simultaneous feedback mechanisms:

1. Decreasing extinction rate, i.e. more taxa accompanies higher alpha diversity, or mean number of taxa in a community and therefore communities become more complex and stable. This is followed by decrease in extinction and more taxa etc.

2. Increasing origination rate, i.e. new taxa facilitate niche construction and newly formed niches can be occupied by the next the generation of taxa.

The latter possibility makes the mechanisms underlying the hyperbolic growth of biodiversity and super-exponential growth of human population even more similar, because the total ecospace of the biota is analogous to the carrying capacity of the Earth in demography. As far as new species can increase ecospace and facilitate opportunities for additional species entering the community, they are analogous to the technological innovators of the demographic models whose inventions increase the carrying capacity of the Earth. The hyperbolic growth of the Phanerozoic biodiverstiy suggests that cooperative interactions between taxa can play an important role in evolution, along with generally accepted competitive interactions. Due to this cooperation, the evolution of biodiversity acquires some features of a self-accelerating process. Macro-evolutionary cooperation reveals itself in increasing stability of communities that arises from alpha diversity growth and in the ability of species to facilitate opportunities for additional species entering the community. (Markov, Korotayev, 2008)

Based on the analogy with different evolutionary processes we can conclude that human population is a dynamical system with non-linear positive feedback occurring between technological advancement and the number of individuals in population. Taking these observational and theoretical assumptions into account we concluded that deterministic super-exponential model is adequate to be considered.

The theoretical grounds for this approach lies in the idea of accelerating returns between human population and technological progress. However, the constraints of the Earth's human carrying capacity are just as real as ever before.

The history of Easter Island provides a case study of human choices and natural constraint in a more comprehensible scale. In ecology, Easter Island (aka Rapi Nui) is of great interest, as one of the many civilizations that undercut their own ecological foundations. The island is one of the most isolated bits of land on Earth, it lies 3,510 km west of continental Chile at its nearest point. Between 1200 and 1500 A.D., the island was inhabited by over 10,000 people and had a rather sophisticated and technologically advanced society. During this time, inhabitants used large boats for fishing and navigation, constructed numerous buildings and built many of the large statues, known as *moai*, for which the island is now best known. By the late 18th century, the population had dropped to 2,000 and almost all elements of the previous society were completely wiped out. However, the details of the population history of the island is full of uncertainties and most of the prehistoric data is based on archaeological excavations. There are various hypotheses on the causes of the abrupt decline of the Rapi Nui culture, one of which the inhabitants overused the resources to the point of starvation and the island's human population decreased drastically. Basener and Ross (2005) devised a mathematical model to describe the dynamics of the population on Eastern island. The model also exhibits finite-time singularity and is claimed to match much more accurately with archaeological data. They proved that if the population uses resources too quickly relative to the rate at which the resources replenish themselves, then the population will increase and then disappear in finite time.

Johansen and Sornette showed that an estimation of the critical time is uncertain due to large fluctuations in the behaviour of the system around the simple model described by Eq. (9). They assumed the presence of log-periodic oscillations in the dynamics of the human population, the GDP of the world, and other types of financial indices. (Johansen, Sornette, 2001)

It is evident that there are two key issues in modelling human population dynamics. How to model the environment of human population dynamics and understand the role of technological progress in the relationship between the environment and the population. The analysis of the previous work done one modelling human population dynamics unveils the crucial role that the technological progress is playing in shaping the evolution of human population and its dynamics. Understanding the significance and meaning of technological progress in respect of demography is one of the key issue in human population dynamics. It has became evident that more analysis on case studies and overall research on linking population dynamics and technological progress is needed.

## 2.3. Interpretating the Demographic Singularity

Finite-time singularity is an inherent feature of the super-exponential model. Ultimately, it is the direct consequence of the accelerating growth rate.

Mathematical singularities are common in many well-established models of natural systems. For example, singularities appear in Euler equations of inviscid fluids. The formation of black holes described by the equations of General Relativity, as well as the beginning of space-time in cosmological models are marked by mathematical singularities. This type of critical behaviour figures extensively in the physics of phase transitions, such as in the Ising model for the magnets or vertex models for ice formation. Mathematical models for aggregating micro-organisms to form fruiting bodies is just another example from biophysics that exhibit critical behavior. Singularities are often used to mark a rupture or a material failure, earthquakes, stock market crashes, and epileptic seizures. (Johansen, Sornette, 2001)

It is clear that we cannot literally interpreted a mathematical singularity as a physical reality. Infinite population size cannot be sustained on Earth. We could argument that the singularity can be seen as a breakdown in our way of mapping and describing the physical reality, rather than a property of physical reality itself. For example, when we look at the lines of latitude and longitude that we employ to map positions on the surface of the Earth. As we move towards the poles on a geographer's globe, the meridians approach one another, and finally intersect. The map coordinates are singular at the poles. Nevertheless, nothing spectacular happens to the Earth's surface there.

However, there are physicists, like Roger Penrose, who regard the singular beginning of the Universe as an important ingredient in its structure, without which we would miss some of its defining characteristics, Penrose–Hawking singularity theorems illustrate how singularities can be used descriptively and explanationarily.

We could also draw parallels between demographic singularity and phase transitions prevalent in non-equilibrium processes. Analogously to non-equilibrium processes, we cannot easily define the relation between the state the population was prior to the critical phase transition and after the occurrence of the singularity. We cannot simply consider the population to remain the same population. Withing the framework of non-equilibrium processes, we can also assume the shift to be strongly irreversible, i.e. the population cannot return to the state prior to demographic singularity. Consequently, it is an abrupt and complete qualitative transition after which the system can no longer be observed, i.e. the extrapolations of the current model would not be appropriate.

The intent of this discussion is to argument that we cannot understand the nature of demographic singularity in a well-described scientific way. The finite-time singularity issued forth by a super-exponential growth model for the human population is a theoretical lacuna that could only be a subject of speculation. Becomingly, there are a plethora of speculations spread around on this topic in the literature.

In pursuance of the "gloomy scenario" the humanity will enter a severe recession fed by the irreversible and immense climate change, in the spirit of the analogy proposed by Hern (1993) between the human species and the proliferating cancer cells. This view differs dramatically from the ubiquitous projections of demography proper which state that human population will stabilize around some specific population size and the population growth would roughly be equal to zero (Lutz *et al.* 2004; UN, 1999). Although, in the latter case, population growth would be characterized by approaching the limit of the carrying capacity, and the population growth would resemble logistic growth. It should also be noted that the "gloomy scenario" is supported by many authorities in science, including the Union of Concerned Scientists, comprising of 99 Nobel Prize winners (Johansen, Sornette, 2001). Furthermore, it has also been contemplated that in case of the "gloomy scenario" systematic development of terrorism and the segregation of mankind into groups, with the minority of privileged elite controlling the rest (Johansen, Sornette, 2001).

On a more optimistic note, it may be argued that humanity will transcend to an eco-friendly society in perfect harmony with its ecological ambience (Johansen, Sornette, 2001). Optimistic views are often developed alongside with the technological positivism. A number of researchers (Chalmers, 2010; Good, 1965; Kurzweil, 2005; Moravec, 1999; Sandberg, 2009; Solomonoff, 1985; Vinge, 1993; Baum, Goertzel, 2010) have argued that in the present century, humanity will develop Artificial Intelligence (AI) programs capable of exceeding human performance in almost every field, including AI research, and that this will greatly accelerate technological progress and ultimately design their successors. That hypothetical event has been described as an *intelligence explosion* or *technological singularity*. The supporters of the idea of technological singularity note that it is difficult or impossible for present-day humans to predict what a post-singularity world would be like, due to the incomprehensible potential of super-intelligent entities.

Chalmers (2010) argues that even if it is technically feasible for humanity to produce a *technological singularity*, it may not be carried out due to the high risks involved. Numerous authors have elaborated on different risks posed by developing AI, including human extinction (Bostrom, 2002; Chalmers, 2010; Friedman, 2008; Hall, 2007; Kurzweil, 2005; Moravec, 1999; Posner, 2004; Yudkowsky, 2008). In addition to anthropomorphic sentiments of AI rebelliousness, a more rigorous argument relies on the sheer power of such entities. Predicting and controlling AI systems operating at super-humanly fast timescales would need decision algorithms with very general applicability. However, irregardless of the risks, the lead in AI technology may well translate into overwhelming military power.

Other optimistic conjectures (Romer, 1990; Kapitza, 2009) include the transition to a knowledge based society, that would embody the balanced symbiosis with the population and the Earth's resources. According to this scenario, the future will not be dominated by food, energy, hardware, or other resources, but by the software produced by the society, and education industry would become a major sector of the economy. Establishment of new social norms and values would follow consequently. In such a society intellectual, artistic and humanistic values would replace the quest for material well-being. The economy of a knowledge based society would differ significantly from the classical notion of economy. It has also been suggested that these type of societies have already been observed to exist in the history (Diamond, 1997).

Other possible scenarios include the continuation or even further acceleration of the growth provided by some new discoveries and advances, for example the colonization of other planets. The cascade of abrupt changes that will take place can also also trigger the evolution of some mutated form of humans who are more endurant to the extreme conditions.

Kapitza (2009) has pointed out that the absence of long-term predictions and the rate at which the abrupt changes would take place, leaves no time for political decisions to be worked out and properly implemented. The lack of social planning leads to the loss of governance with law and order, corruption, criminality and a rise of irrationality, seen on many levels of society and political structures of the world.

In conclusion, the physical metaphor of a demographic singularity is very difficult to understand. The super-exponential model should be used to describe the dynamics where it fits data well or seems like a reasonable prediction - the meaning of singularity should be treated as a black box.

# 3. STOCHASTIC SUPER-EXPONENTIAL MODEL

## 3.1. Theoretical Background of the Stochastic Model

All biological populations fluctuate in a stochastic fashion. There are ample of interesting phenomena known in population ecology that arise from the inherent variability from the dynamics create increased risk that do not exist in idealized deterministic models, e.g. Allee effect in fluctuating populations. The dynamics of every population has both deterministic (predictable from initial conditions) and stochastic components that operate simultaneously. There are basic forms of stochasticity that play an important role in modelling population dynamics: demographic stochasticity, environmental stochasticity, measurement error, and uncertainty arising from the differences between real world and the theoretical assumptions that make up mathematical models:

1. Demographic stochasticity refers to probabilistic events associated with the individual mortality and reproduction, which are usually conceived as being independent among individuals. In any given time period, an individual either dies or survives with a certain probability. Individuals also have a probability distribution of offspring produced per unit time. Because demographic stochasticity operates independently among individuals, it tends to average out in large populations and has a greater impact on small populations.

2. Environmental variability affects local and global human populations through wide range of factors, such as the weather, epidemics, accidents, crop diseases and pests, volcanic eruptions, tsunamis, tornadoes, the El Ninon Southern Oscillations in the Pacific Ocean, wildfires, genetic variability in viruses and other microbes, and international financial, political and cultural arrangements. The impact of environmental stochasticity is roughly the same for small and large populations. Stochastic modelling provides important contributions in both pure and applied ecology.

3. Measurement error in estimates of population size is also a definite source of uncertainty. There are many things which the population dynamics depend upon which are hard or impossible to specify. As was already stated in Chapter 1, the Earth's population probably could not be known with an uncertainty of less than around 20 percent before the middle of the eighteen century (Coale, 1974). For populations in which a complete and accurate census is available, measurement error can be ignored.

4. Another reason to use stochastic models stems from the fact that there is always information that is left out when a model is made. These omissions may simply be ignored or included in the sense that there is uncertainty and stochasticity in dynamics of the

> epiphenomenon, the population dynamics. These uncertainties may be due to differences in dynamics in the sub-processes that are not taken into account. The underlying problem is that we are not able to capture the whole complexity of the ecological situation. Models are idealized versions of the real world or as George Box put it, "All models are false, but some models are useful" (Kokko, 2007).

In order to implement stochastic variability to the growth rate we can express the annual growth rate as the annual mean $\bar{r}$, plus the annual deviation $\xi(t)$ from the mean, with an intensity $\sqrt{D'}$

$$r = \bar{r} + \sqrt{D'}\xi(t), \tag{12}$$

where $\xi$ is a Gaussian white noise with the properties $\langle \xi(t) \rangle = 0$, $\langle \xi(t)\xi(t') \rangle = \delta(t-t')$ and $D'$ denotes the intensity of the noise. We have assumed the noise $\xi$ to be Gaussian white noise interpreted a la Stratonovich. This assumption is due to the fact that environmental changes are consequences to a large number of independent factors and the time-scale of the demographic mechanisms that determine the growth rate is much larger than the time-scale of the environmental perturbations (Livi-Bacci, 2006). Moreover, the aim of this model is to describe the dynamics of human population for the past 10 000 years. Therefore, we can expect that the stochastic events that effect human population growth rate in long-run are not correlated. This assumption is expected to hold since it is hardly conceivable that the population size $X(t)$ would depend in a straightforward manner from centuries past events.

## 3.2. Langevin Equation for World Population

Applying the stochastic growth rate (12) on the super-exponential model (9), yields the nonlinear Langevin equation

$$\frac{dX}{dt} = x^{\alpha+1}\left[\bar{r} + \sqrt{D'}\xi(t)\right], \tag{13}$$

which is defined as a stochastic super-exponential growth model for human population. The noise in Eq. (13) can be an attribute of all variability categories mentioned in the previous section, i.e. demographic, environmental, measurement error and error arising from the differences between mathematical presumptions and the elusive nature of ecological complexity. Subsequently, the dash on the growth rate is omitted, i.e. $r \equiv \bar{r}$. The fairly simple structure of the nonlinear Langevin equation (13) enables it to be linearized by means of the transformation

$$y = x^{-\alpha}, \tag{14}$$

which yields

$$\frac{dY}{dt} = -\alpha r + \sqrt{D}\xi(t), \quad \alpha > 1, \tag{15}$$

where the noise intensity $D = \alpha^2 D'$. The state $y = \infty$ (the natural extinction of the population, i.e. the state $x = 0$) is interpreted as a reflecting boundary. Prima facie, it may seem as a tedious premiss, but it will transpire in a later discussion that the population will never reach its reflecting boundary. The initial condition for Eq. (15) is defined as $Y(0) = y_0$, which is interpreted as an absorbing boundary. The state $y = 0$ (the population size $x = \infty$) denotes the demographic singularity and is interpreted as an absorbing boundary. Recall that the demographic singularity ($x = \infty$) represents a catastrophic event for the population and the two main possible scenarios include:

- the extinction of the population;
- transformation into a new population differing qualitatively from the current one.

## 3.3. Conditional Probability Density

In order to understand the temporal behaviour of the population as a stochastic process, we examine the conditional probability density function (CPDE) $P(x,t|x_0,0)$ of the population size $X(t)$. By means of the Stratonovich calculus, the Langevin equation (15) leads to the following Fokker-Planck equation (FPE)

$$\frac{\partial \tilde{P}(y,t|y_0,0)}{\partial t} = \alpha r \frac{\partial \tilde{P}(y,t|y_0,0)}{\partial y} + \frac{D}{2}\frac{\partial^2 \tilde{P}(y,t|y_0,0)}{\partial y^2}, \tag{16}$$

where $\tilde{P}(y,t|y_0,0)$ denotes the conditional probability density of $Y(t)$ with the initial condition $\tilde{P}(y,0|y_0,0) = \delta(y-y_0)$ and $y \geq 0$. The biologically meaningful boundary condition for the probability density $\tilde{P}(y,t|y_0,0)$ is

$$\tilde{P}(y,t|y_0,t)\mid_{y=0} = 0. \tag{17}$$

In solving the Fokker-Planck equation (16) we searched for a solution in the general form of

$$\tilde{P}(y,t|y_0,0) = exp\left[-\tfrac{\alpha r}{D}(y - y_0 + \tfrac{\alpha r}{2}t)\right] \cdot U(y,t|y_0,0), \tag{18}$$

where function $U(y,t|y_0,0)$ satisfies the diffusion equation

$$\frac{\partial U(y,t|y_0,0)}{\partial t} = \frac{D}{2}\frac{\partial^2 U(y,t|y_0,0)}{\partial y^2}, \tag{19}$$

with absorbing boundary $U(y,t|y_0,t)\mid_{y=0} = 0$ and initial condition $U(y,0|y_0,0) = \delta(y-y_0)$. This well-known equation was solved by means of method of images (Kampen, 1992) and yields the following solution

$$U(y,t|y_0,0) = \frac{1}{\sqrt{2\pi D t}}\left\{exp\left[-\frac{(y-y_0)^2}{2Dt}\right] - exp\left[-\frac{(y+y_0)^2}{2Dt}\right]\right\}. \tag{20}$$

Finally, from Eq. (18) and (20) we obtain the solution that satisfies the Fokker-Planck equation (16)

$$\tilde{P}(y,t|y_0,0) = \frac{1}{\sqrt{2\pi Dt}} \left\{ exp\left[ -\frac{(y-y_0+\alpha rt)^2}{2Dt} \right] - exp\left( \frac{2y_0\alpha r}{D} \right) exp\left[ -\frac{(y+y_0+\alpha rt)^2}{2Dt} \right] \right\}. \tag{21}$$

After transforming back to the original variable $x$, we obtain the conditional probability density function for human population size

$$P(x,t|x_0,0) = \frac{\alpha x^{-(\alpha+1)}}{\sqrt{2\pi Dt}} \left\{ exp\left[ -\frac{(x^{-\alpha}-x_0^{-\alpha}+\alpha rt)^2}{2Dt} \right] - exp\left( \frac{2x_0^{-\alpha}\alpha r}{D} \right) exp\left[ -\frac{(x^{-\alpha}+x_0^{-\alpha}+\alpha rt)^2}{2Dt} \right] \right\} . \tag{22}$$

It follows from the probability density (22) that the population will not become extinct in a natural manner, i.e. $P(x,t|x_0,0)\ |_{x=0} = 0$. The characteristical behaviour of the conditional probability density function $P(x,t|x_0,0)$ has been illustrated on Fig. (11)[2].

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[2] There are two yet unspecified but important quantities introduced in the footer of this figure and all the following figures: the mean lifetime of the population $\langle T \rangle$ and the population's prospective lifetime $t_p$. In section 3.6, these quantities will be introduced and more thoroughly elaborated on.

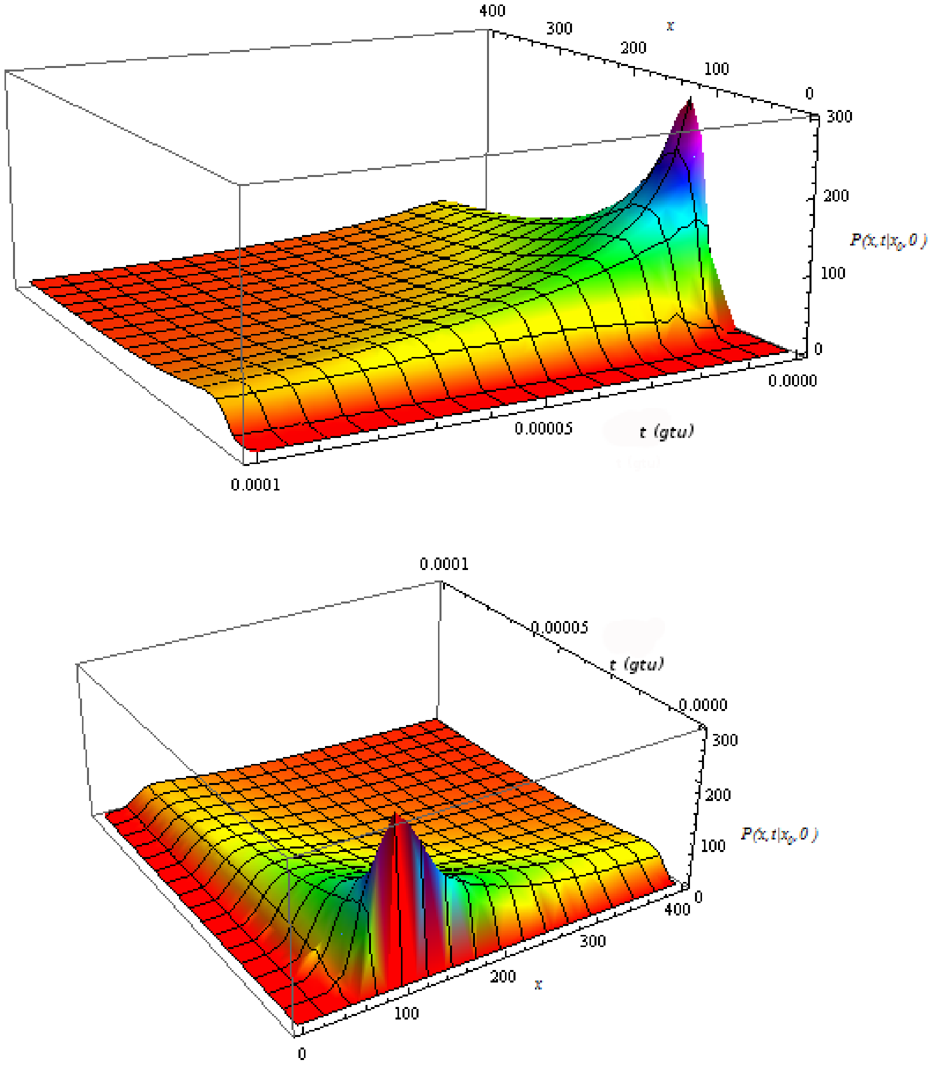

Figure 11: The conditional probability density function $P(x,t|x_0,0)$. The 3D representations of the conditional probability density $P(x,t|x_0,0)$ is presented from two different angles with parameter values: the initial population size $x_0 = 100$, the growth rate $r = 0.01\,(gtu)^{-1}$, the noise intensity $D = 0.5\,(gtu)^{-1/2}$ and the technological parameter $\alpha = 1.2$. The mean value of the population lifetime $\langle T \rangle = 0.33\,(gtu)$ and the most probable value of the lifetime $t_p = 0.000011\,(gtu)$.

On Fig. (11) and hereafter we have used general time unit ($gtu$), which stands for some unspecified time unit. For our purposes it seemed useful to use such abstract time values because we are not fitting the actual data with this theoretical model. Since we are interested in the influence of noise and its interplay with nonlinear dynamics within our system we are not constrained by the choice of units.

The probability distribution together with the set of possible states fully define the stochastic process. However, number of additional concepts are often used to make use of the stochastic model in real physical systems. In the succeeding chapters we will introduce these concepts and their function in population dynamics.

## 3.4. The Mean Value

The exact solution for the mean value of the population size can be calculated subsequently from the probability distribution, defined as

$$\langle X(t)\rangle = \int_0^\infty xP(x,t|x_0,0)dx. \tag{23}$$

It is useful to transform the afore-presented integrand into the variable $y$ notation

$$\langle X(t)\rangle = \left\langle [Y(t)]^{-1/\alpha} \right\rangle = \int_0^\infty y^{-1/\alpha}\tilde{P}(y,t|y_0,0)dy, \tag{24}$$

where the variable $y$ and probability density $\tilde{P}(y,t|y_0,0)$ are given by Eq. (14) and Eq. (22), respectively. After comparatively simple but voluminous calculations, the following formula for the mean value of population size was obtained

$$\begin{array}{c} \langle X(t)\rangle = \frac{\Gamma\left(1-\frac{1}{\alpha}\right)}{\sqrt{2\pi}(Dt)^{\frac{1}{2\alpha}}} \left\{ exp\left[-\frac{(\alpha rt - x_0^{-\alpha})^2}{4Dt}\right] D_{\left(\frac{1}{\alpha}-1\right)}\left(\frac{\alpha rt - x_0^{-\alpha}}{\sqrt{Dt}}\right) \right. \\ \left. -exp\left(\frac{2\alpha r x_0^{-\alpha}}{D}\right) exp\left[-\frac{(\alpha rt + x_0^{-\alpha})^2}{4Dt}\right] D_{\left(\frac{1}{\alpha}-1\right)}\left(\frac{\alpha rt + x_0^{-\alpha}}{\sqrt{Dt}}\right) \right\} \end{array}, \tag{25}$$

where $\Gamma(x)$ is the gamma function and $D_a(b)$ is the parabolic cylinder function.

The precise relation of the mean value of the population size $\langle X(t)\rangle$ and the actual observed population size $X(t)$ is perhaps somewhat vague. In present case, we can imagine having a large number of copies of the population, i.e. population ensemble (in the ideal situation we should have an infinite amount of them), each having some fluctuating population size. The mean value $\langle X(t)\rangle$ is a good estimate for the average populations size in the ensemble, if we could conduct censuses on all of them instantaneously at each time moment. This "many systems" approach is more supported by the theoreticians among statistical physicists and was coined as an ensemble interpretation. The ensemble interpretation was originally introduced by Gibbs. The trouble with this interpretations is that we only have one human population and we can only conduct censuses on that population and we also cannot make instantaneous measurements, but rather integrate our data over some period of time. In statistical physics there is also another way of understanding the relation between the mean value and the actual observed value, which cannot be applied to our system. This is called the "time average" interpretation, which relies on the ergodic hypothesis proposed by Boltzmann. The ergodic hypothesis says that, over long periods of time, the time spent by a population in some region of the phase space of micro-states is proportional to the volume of that region, i.e. all accessible micro-states are equiprobable over a long period of time. For instance, in our discussion, "some region of the phase space of micro-states" would mean "in some range of population sizes". Therefore, all possible population sizes should be equiprobable over a long period of time. However, for our model there is no way that ergodicity would hold. The population has no "stationary" state in our system, which is needed for the "time average" interpretation.

## 3.5. The Second Moment and the Relative Variance

In order to estimate the variance around the mean value of the population size we also calculated the second central moment. The second moment is defined as the integral

$$\left\langle X^2(t)\right\rangle = \left\langle [Y(t)]^{-2/\alpha}\right\rangle = \int_0^\infty y^{-2/\alpha}\tilde{P}(y,t|y_0,0)dy, \tag{26}$$

which converges when $\alpha > 2$ and for the case $1 < \alpha < 2$ we integrated Eq. (26) by parts. For the case $\alpha = 2$ there is a regular singular point, because

$$lim_{\alpha\to 2^+}\int_0^\infty y^{-2/\alpha}\tilde{P}(y,t|y_0,0)dy = lim_{\alpha\to 2^-}\int_0^\infty y^{-2/\alpha}\tilde{P}(y,t|y_0,0)dy.$$

Finally, the second moment for $1 < \alpha$ yields

$$\begin{aligned}\left\langle X^2(t)\right\rangle = \frac{\Gamma\left(1-\frac{2}{\alpha}\right)}{\sqrt{2\pi}(Dt)^{\frac{1}{\alpha}}} exp\left[-\frac{(\alpha rt - x_0^{-\alpha})^2}{4Dt}\right]\Big\{ & D_{\left(\frac{2}{\alpha}-1\right)}\left(\frac{\alpha rt - x_0^{-\alpha}}{\sqrt{Dt}}\right) \\ & - exp\left[\frac{\alpha r x_0^{-\alpha})}{D}\right] D_{\left(\frac{2}{\alpha}-1\right)}\left(\frac{\alpha rt + x_0^{-\alpha}}{\sqrt{Dt}}\right)\Big\}\end{aligned}. \tag{27}$$

We can use the latter equation to calculate the standard deviation of the population size, which is defined as

$$\sigma(X) = \sqrt{\langle X^2(t)\rangle - \langle X(t)\rangle^2}. \tag{28}$$

Subsequently, we also define the relative variance of the population size as

$$\delta(X) = \frac{\sqrt{\langle X^2(t)\rangle - \langle X(t)\rangle^2}}{\langle X(t)\rangle}. \tag{29}$$

We prefer to use relative variance as an estimate of the dispersion around the mean value, because it has the same dimensionality.

## 3.6. Probability Density of a Population's Lifetime, Mean Lifetime and the Prospective Lifetime

We have seen that mean value of population size tends zero eventually. This means that given enough time most of the populations in the population ensemble will reach the absorbing boundary $x = \infty$. Naturally, we want to estimate how long on average does it take a population to reach the absorbing boundary. In this section we derive a formula for estimating the average time it takes for a population to reach the demographic singularity, and its probability density, as well as the population's prospective lifetime.

Lets denote the time interval during which it takes a population to reach the absorbing state from its initial state with $T(x_0)$ and coin it as a *lifetime of the population*. From the Fokker-Planck equation via the Smoluchowski equation differential equations defining the mean lifetime $\langle T(x_0)\rangle$ and the second moment of the lifetime $\left\langle T^2(x_0)\right\rangle$ can be derived

$$\alpha r \frac{\partial \langle T(x_0) \rangle}{\partial x_0} + \frac{D}{2} \frac{\partial^2 \langle T(x_0) \rangle}{\partial x_0^2} = -1, \tag{30}$$

$$\alpha r \frac{\partial \left\langle T^2(x_0) \right\rangle}{\partial x_0} + \frac{D}{2} \frac{\partial^2 \left\langle T^2(x_0) \right\rangle}{\partial x_0^2} = -2 \langle T(x_0) \rangle \tag{31}$$

with boundary conditions for the absorbing boundaries

$$\langle T(x_0) \rangle |_{x_0=\infty} = 0, \tag{32}$$

$$\left\langle T^2(x_0) \right\rangle |_{x_0=\infty} = 0 \tag{33}$$

and reflecting boundaries

$$\frac{d \langle T(x_0) \rangle}{d x_0} |_{x_0=0} = 0, \tag{34}$$

$$\frac{d \left\langle T^2(x_0) \right\rangle}{d x_0} |_{x_0=0} = 0. \tag{35}$$

From Eq. (30) and boundary conditions (32) and (34) it easily follows that

$$\langle T(x_0) \rangle = \frac{1}{x_0^\alpha \alpha r}, \tag{36}$$

which coincides with the the deterministic critical time $t_c$ given by Eq. (11), i.e.

$$\langle T(x_0) \rangle = t_c. \tag{37}$$

From Eq. (31) and boundary conditions (33) and (35) it follows that

$$\left\langle T^2(x_0) \right\rangle = \frac{D}{x_0^\alpha (\alpha r)^3} + \left( \frac{1}{x_0^\alpha \alpha r} \right)^2, \tag{38}$$

The relative variance of the lifetime of the population has the following form

$$\delta(T) = \sqrt{\frac{x_0^\alpha D}{\alpha r}}. \tag{39}$$

Let $P_T(x_0, t)$ denote the probability density of the population's lifetime. Let $a$ mark the absorbing boundary and $r$ identify the reflecting boundary. We can easily reason that the probability density of the time it takes for a random population to reach its absorbing boundary can be written as an integral

$$P_T(x_0, t) = -\int_r^a \frac{\partial}{\partial t} P(x, t | x_0, 0) dx, \tag{40}$$

which has the following form for the current situation

$$P_T\left(x_0,t\right)=-\int\limits_0^{\infty}\frac{\partial}{\partial t}P(x,t|x_0,0)dx, \tag{41}$$

which can also been written in terms of

$$P_T\left(y_0,t\right)=\int\limits_0^{\infty}\frac{\partial}{\partial t}\tilde{P}(y,t|y_0,0)dy. \tag{42}$$

Integral (42) can be easily calculated by means of the Fokker-Planck equation and the probability density of the population's lifetime has the form

$$P_T\left(x_0,t\right)=\frac{x_0^{-\alpha}t^{-3/2}}{\sqrt{2\pi D}}exp\left(-\frac{\left(\alpha rt-x_0^{-\alpha}\right)^2}{2Dt}\right). \tag{43}$$

Next, we define the maximum point of the probability density $P_T\left(x_0,t\right)$, i. e. the time value that corresponds to the maximum of the probability density

$$t_p=\sqrt{\frac{9D^2}{4\left(\alpha r\right)^4}+\left(\frac{1}{x_0^{\alpha}\alpha r}\right)^2}-\frac{3D}{2\left(\alpha r\right)^2}. \tag{44}$$

as the population's prospective lifetime, because it is the most probable value of the population's lifetime. Equation (44) implies that in the absence of noise $\langle T\rangle=t_p$ and therefore $t_c=t_p$. Anywhere in the region $D\in(0,\infty)$, however, the ratio $0<t_p/\langle T\rangle<1$, which means that the population's prospective lifetime is always shorter than the mean lifetime of the system. In fact $t_p/\langle T\rangle$decays rapidly as a power law when $D\to\infty$. The same behaviour holds for others parameters as well, $t_p/\langle T\rangle$ decays rapidly as a power law when $\alpha\to\infty$ or $r\to\infty$, as long as, $D>0$. We can conclude that if noise is introduced to the system the population's prospective lifetime is significantly smaller than the mean lifetime of the population $\langle T\rangle$.

# 4. Environmental Noise Induced Effects

In this part we analyze the results obtained from the previous part and interpret the results in context of the human population. In this section we observe the interaction between nonlinear deterministic components of the system and the stochastic factors inherent within the system. The goal in synthesizing these approaches is to understand how population fluctuations arise from the interplay of noise and nonlinear dynamics. The comparable importance of deterministic and stochastic forces makes ecological dynamics unique. Indeed, the interaction between noise and nonlinear determinism in ecological dynamics adds an extra level of complexity compared with the largely stochastic dynamics like the Brownian particles or the largely deterministic dynamics of many simple physical processes.

## 4.1. Mean value of the Population Size

Figures (12), (13) and (14) illustrate the effect of the technological parameter, the noise intensity and the growth rate on the mean value of the population size. These figures also illustrate the general characteristics of the mean value of the population size respectively. Most significantly we can see that the mean value has a maximum.

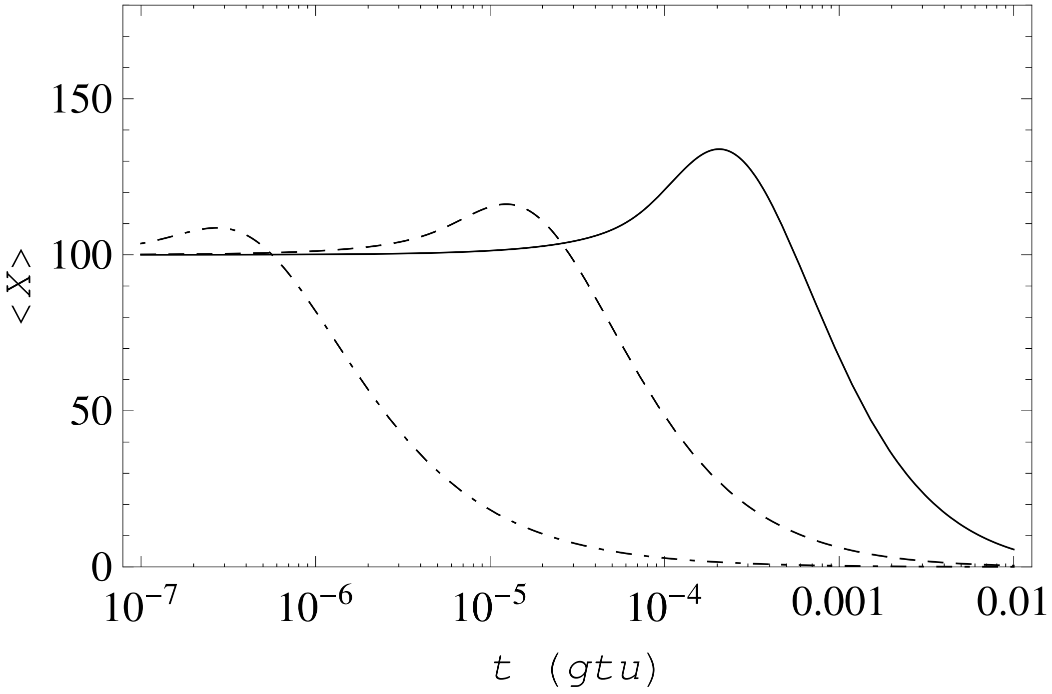

Figure 12: The mean value of population size $\langle X(t) \rangle$ as a function of time $t$ at various values of the noise intensity $\alpha$ is illustrated. The growth rate $r = 2\,(gtu)^{-1}$, the noise intensity $D = 0.1\,(gtu)^{-1/2}$ and the initial population size $x_0 = 100$. Solid line: $\alpha_1 = 1.01$, with the mean lifetime of the population $\langle T_1 \rangle = 0.0047\,(gtu)$ and the population's prospective lifetime $t_{p1} = 0.00030\,(gtu)$; dashed line: $\alpha_2 = 1.3$, with the mean lifetime of the population $\langle T_2 \rangle = 0.00097\,(gtu)$ and the population's prospective lifetime $t_{p2} = 0.000021\,(gtu)$; dashed-dotted line: $\alpha_3 = 1.7$, with the mean lifetime of the population $\langle T_3 \rangle = 0.00012\,(gtu)$ and the population's prospective lifetime $t_{p3} = 5.3 \cdot 10^{-7}\,(gtu)$.

Figure (12) plots the mean value $\langle X(t) \rangle$, as a function of the parameter $t$, at different values of the technological parameter $\alpha$. Similarly to Fig. (14) the maximum of the mean value appears. Notably, the lower values of the technological parameter corresponds to the upper maximum of the mean value of the population size.

Varying the noise intensity $D$ also has a fairly significant effect on the system. Figure (13) illustrates the nature of the impact of environmental noise on the mean value of the population size. The upper maxima of the mean value $\langle X(t) \rangle$ are in accord with lower values of the noise intensity $D$. The latter notion is consistent with the intuitively well-understood assumption that as the environmental noise fades, the stochastic behaviour of a system approaches deterministic behaviour. The global maximizer of the mean value $\langle X(t) \rangle$ is lower for higher values of the noise intensity $D$. Therefore, transitions from the global maxima to the state $x = 0$ are relatively more abrupt if the noise intensity $D$ is lower.

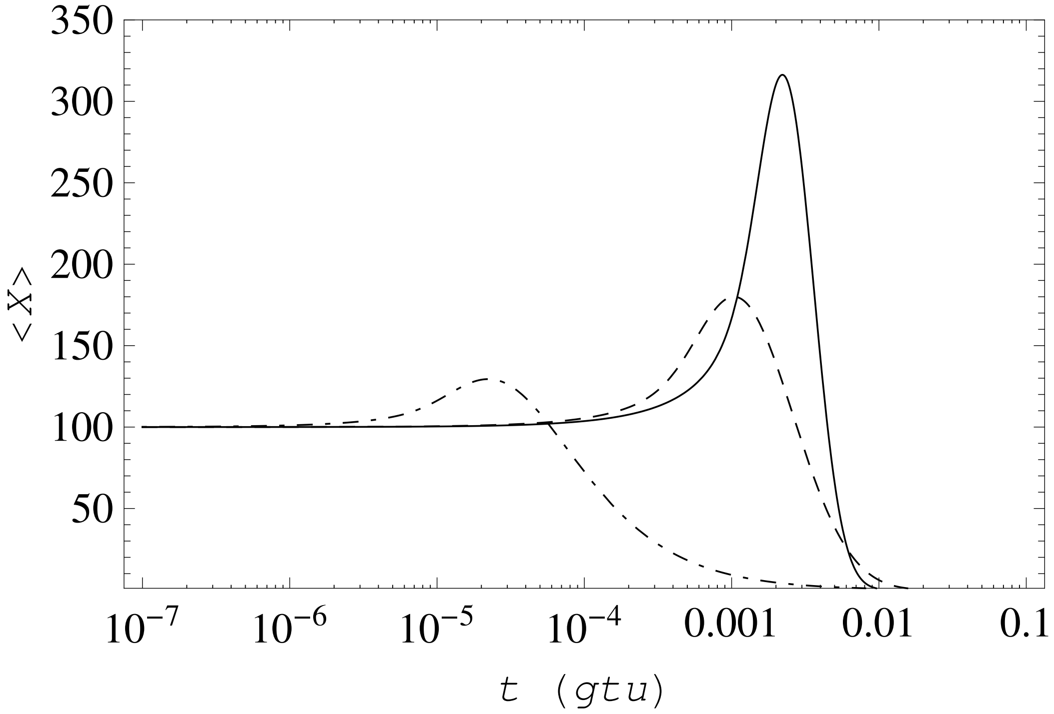

Figure 13: The mean value of population size $\langle X(t)\rangle$ as a function of time $t$ at various values of the noise intensity $D$ is illustrated. The growth rate $r = 3.0\,(gtu)^{-1}$, the technological parameter $\alpha = 1.00001$ and the initial population size $x_0 = 100$. Solid line: $D_1 = 0.005\,(gtu)^{-1/2}$, with the population's prospective lifetime $t_{p1} = 0.0026\,(gtu)$; dashed line: $D_2 = 0.02\,(gtu)^{-1/2}$, with the population's prospective lifetime $t_{p2} = 0.0014\,(gtu)$; dashed-dotted line: $D_3 = 1.0\,(gtu)^{-1/2}$, with the population's prospective lifetime $t_{p3} = 0.000033\,(gtu)$. The mean lifetime of the population is same for all values of the noise density: $\langle T\rangle = 0.0033\,(gtu)$.

The temporal behaviour of the mean value $\langle X(t)\rangle$ at different values of the growth rate $r$ for a fixed technological parameter $\alpha$, the noise intensity $D$, and initial condition $x_0$ is shown on Fig. (14). We observe that the higher values of the growth rate $r$ bring along higher maxima of the mean value of the population size$\langle X(t)\rangle$.

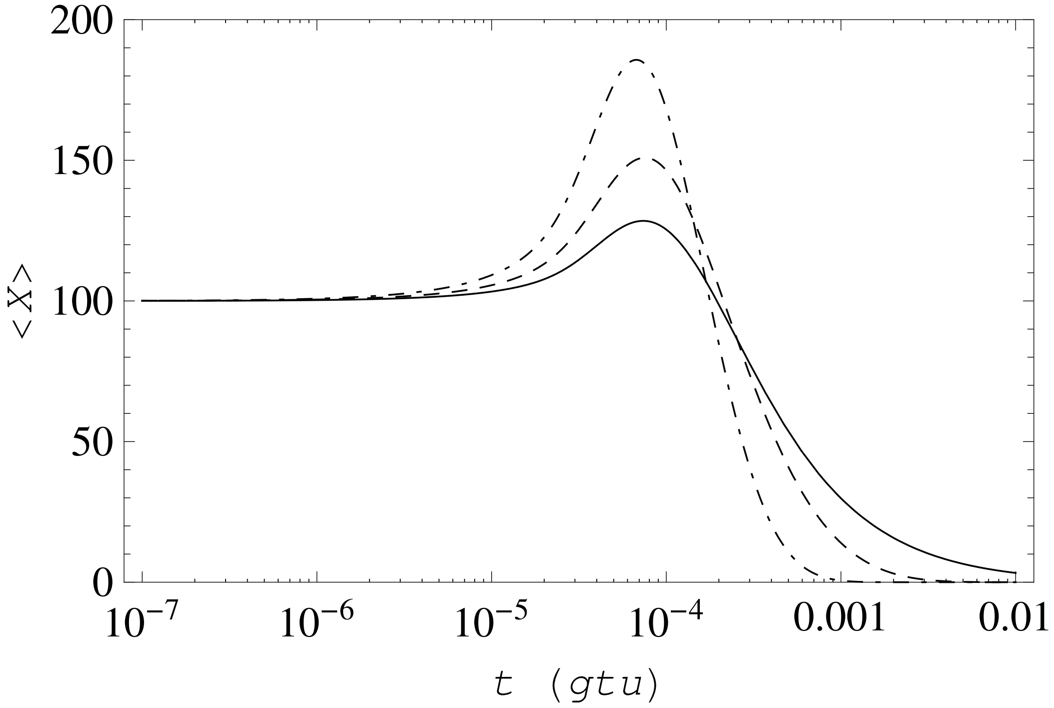

Figure 14: The mean value of population size $\langle X(t)\rangle$ as a function of time $t$ at various values of the growth rate $r$ is illustrated. The technological parameter $\alpha = 1.0001$, the noise intensity $D = 0.3\,(gtu)^{-1/2}$ and the initial population size $x_0 = 100$. Solid line: $r_1 = 10^{-5}\,(gtu)^{-1}$, with corresponding mean lifetime of the population $\langle T_1\rangle = 1000\,(gtu)$ and the population's prospective lifetime $t_{p1} = 0.00011\,(gtu)$; dashed line: $r_2 = 20\,(gtu)^{-1}$, with corresponding mean lifetime of the population $\langle T_2\rangle = 0.00050\,(gtu)$ and the population's prospective lifetime $t_{p2} = 0.00011\,(gtu)$; dashed-dotted line: $r_3 = 50\,(gtu)^{-1}$, with corresponding mean lifetime of the population $\langle T_3\rangle = 0.00020\,(gtu)$ and the population's prospective lifetime $t_{p3} = 0.000089\,(gtu)$.

## 4.2. Relative Variance of the Population Size

The standard deviation of the population size is illustrated on Fig. (15). From the illustrations, we can infer that the standard deviation exceeds the mean value at a certain time moment before the population's prospective lifetime $t_p$. We can also detect the presence of time delay between the maximal values of mean value and the standard deviation. The maxima of the standard deviation exceed the maxima of the mean value and they also reach to the maximal values with the time delay compared with the corresponding mean value. The time delay is larger for greater noise intensity $D$.

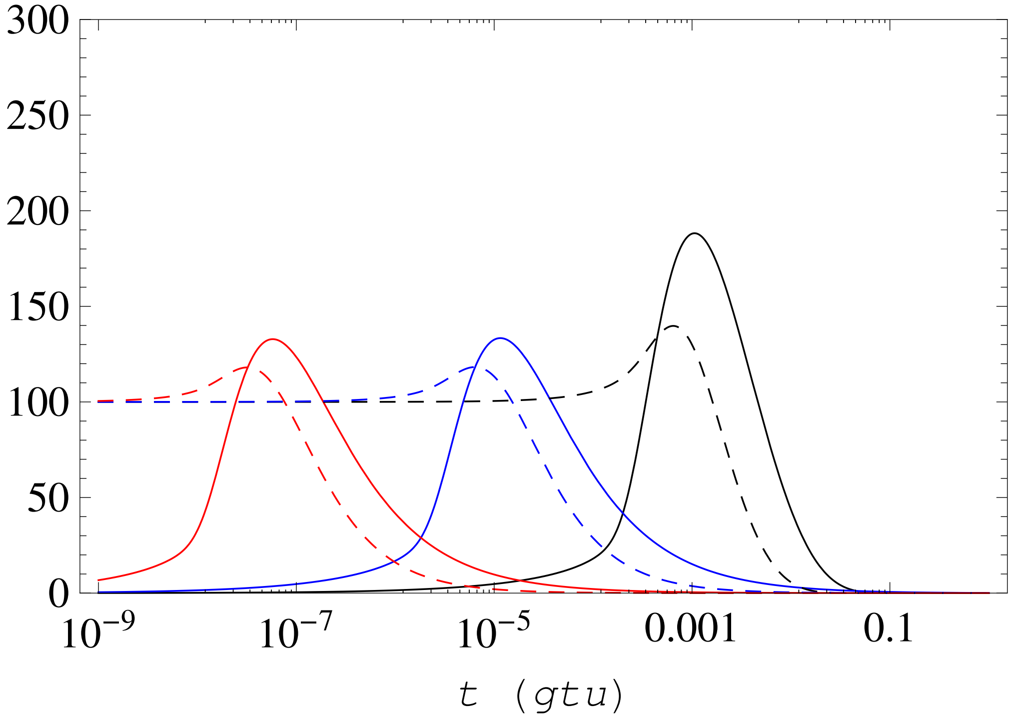

Figure 15: Mean value and standard deviation at different values of noise intensity $D$ is illustrated. The growth rate $r = 1.0\,(gtu)^{-1}$, the technological parameter $\alpha = 1.2$ and the initial population size $x_0 = 100$. The standard deviations are presented as solid lines and the mean values as dashed lines. Black: $D_1 = 0.0050\,(gtu)^{-1/2}$, with the population's prospective lifetime $t_{p1} = 0.00097\,(gtu)$; blue: $D_2 = 0.50\,(gtu)^{-1/2}$, with the population's prospective lifetime $t_{p2} = 0.000011\,(gtu)$; red: $D_3 = 100\,(gtu)^{-1/2}$, with the population's prospective lifetime $t_{p3} = 5.3 \cdot 10^{-8}\,(gtu)$. The population's mean lifetime for these parameters is $\langle T \rangle = 0.0033\,(gtu)$.

Figures (16), (17) and (18) illustrate the effect of the technological parameter, the noise intensity and the growth rate on the relative variance of the population size $\delta(X)$. These figures also illustrate the general characteristics of the relative variance of the population size, respectively. Most significantly, we can observe relatively fast divergence of the relative variance with time. However, no asymptotic behaviour was detected. Remarkably, the relative variance is larger than 1, for populations at time $t > t_p$. This property is due to the fact that CPDF of the population size is strongly asymmetric.

The temporal behaviour of the relative variance of the population size $\delta(X)$ at different values of the technological parameter $\alpha$ for the fixed growth rate $r$, the noise intensity $D$, and initial condition $x_0$ is shown on Fig. (16). As the time $t$ increases, the relative variance $\delta(X)$ diverges. Higher values of the technological parameter are accompanied by faster divergences by the relative variance $\delta(X)$. This is due to the fact that standard deviation exceeds mean value before the population's prospective lifetime. The divergence is much more abrupt for smaller technological parameter $\alpha$.

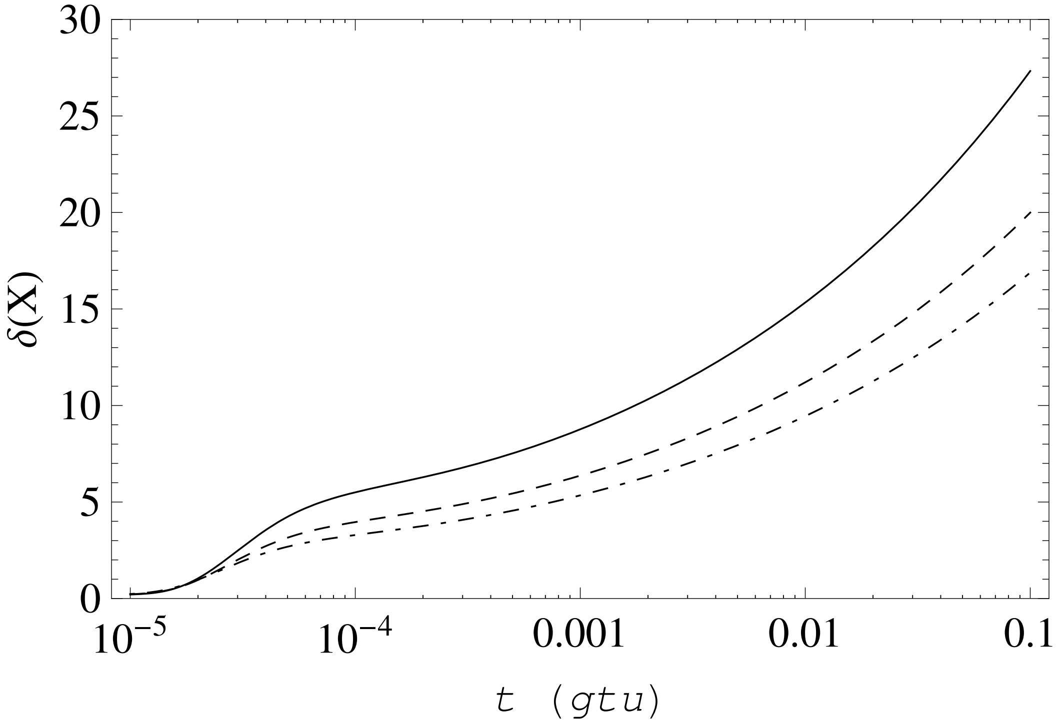

Figure 16: The relative variance of population size $\delta(t)$ as a function of time $t$ at various values of the technological parameter $\alpha$ is illustrated. The parameter values: growth rate $r = 0.01\,(gtu)^{-1}$, noise intensity $D = 0.30\,(gtu)^{-1/2}$ and the initial population size $x_0 = 100$. Solid line: $\alpha_1 = 1.01$, with with corresponding mean lifetime of the population $\langle T_1 \rangle = 0.95\,(gtu)$ and the population's prospective lifetime $t_{p1} = 0.00010\,(gtu)$; dashed line: $\alpha_2 = 1.02$, with corresponding mean lifetime of the population $\langle T_2 \rangle = 0.89\,(gtu)$ and the population's prospective lifetime $t_{p2} = 0.000092\,(gtu)$; dashed-dotted line: $\alpha_3 = 1.03$, with corresponding mean lifetime of the population $\langle T_3 \rangle = 0.85\,(gtu)$ and the population's prospective lifetime $t_{p3} = 0.000084\,(gtu)$.

The temporal behaviour of the relative variance of the population size $\delta(X)$ at different values of the noise intensity $D$ for the fixed growth rate $r$, the technological parameter $\alpha$, and initial condition $x_0$ is shown on Fig. (17). It is observable that the higher values of the noise intensity are accompanied by faster divergences by the relative variance $\delta(X)$.

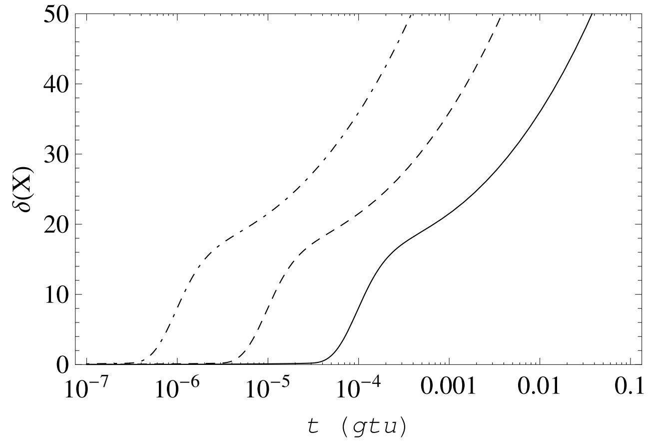

Figure 17: The relative variance of population size $\delta(t)$ as a function of time $t$ at various values of the noise intensity $D$ is illustrated. The growth rate $r = 0.01\,(gtu)^{-1}$, the technological parameter $\alpha = 1.001$ and the initial population size $x_0 = 100$. Solid line: $D_1 = 0.1\,(gtu)^{-1/2}$ and the population's prospective lifetime $t_{p1} = 0.00033\,(gtu)$; dashed line: $D_2 = 1.0\,(gtu)^{-1/2}$, and the population's prospective lifetime $t_{p2} = 0.000033\,(gtu)$; dashed-dotted line: $D_3 = 10\,(gtu)^{-1/2}$ and the population's prospective lifetime $t_{p3} = 3.3 \cdot 10^{-6}\,(gtu)$. The mean lifetime of the population for these parameters is $\langle T \rangle = 1.0\,(gtu)$.

The temporal behaviour of the relative variance of the population size $\delta(X)$ at different values of the growth rate $r$ for the fixed noise intensity $D$, the technological parameter $\alpha$, and initial condition $x_0$ is shown on Fig. (18). It is observable that the higher values of the growth rate are accompanied by faster divergences by the relative variance $\delta(X)$.

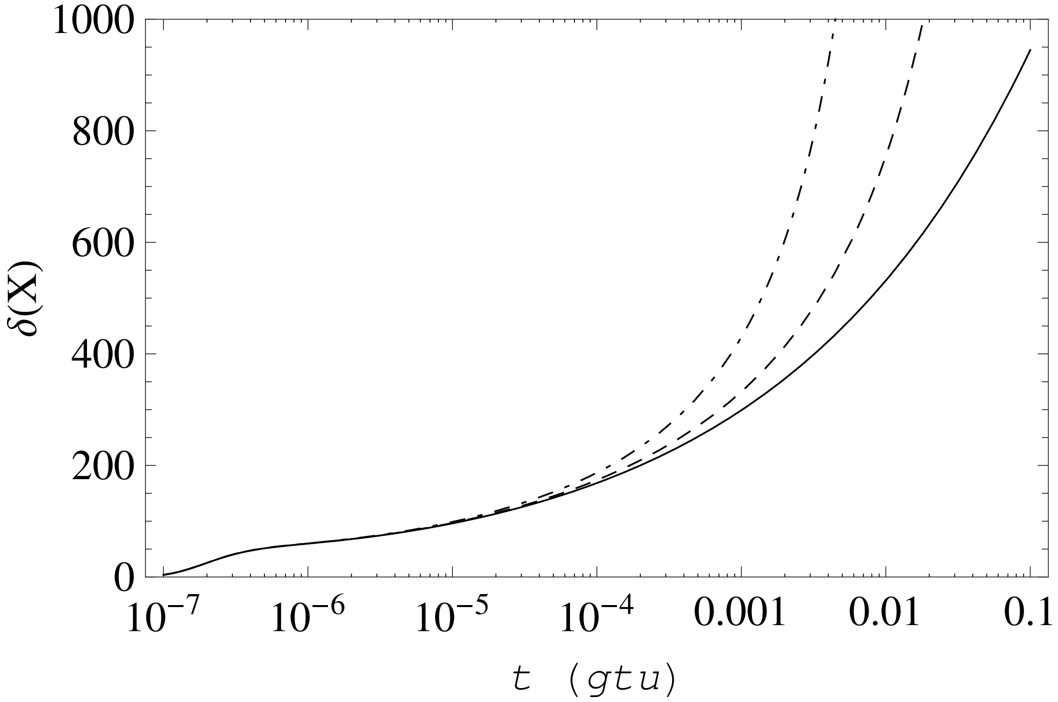

Figure 18: The relative variance of population size $\delta(t)$ as a function of time $t$ at various values of the growth rate $r$ is illustrated. The technological parameter $\alpha = 1.0001$, the noise intensity $D = 50\,(gtu)^{-1/2}$ and the initial population size $x_0 = 100$. Solid line: $r_1 = 0.0001\,(gtu)^{-1}$, with with corresponding mean lifetime of the population $\langle T_1 \rangle = 100\,(gtu)$ and the population's prospective lifetime $t_{p1} = 9.5 \cdot 10^{-7}\,(gtu)$; dashed line: $r_2 = 30\,(gtu)^{-1}$, with with corresponding mean lifetime of the population $\langle T_2 \rangle = 0.00033\,(gtu)$ and the population's prospective lifetime $t_{p2} = 6.7 \cdot 10^{-7}\,(gtu)$; dashed-dotted line: $r_3 = 100\,(gtu)^{-1}$, with with corresponding mean lifetime of the population $\langle T_3 \rangle = 0.00001\,(gtu)$ and the population's prospective lifetime $t_{p3} = 6.7 \cdot 10^{-7}\,(gtu)$.

## 4.3. The Mean Lifetime of the Population and its Probability Distribution and Population's Prospective Lifetime

We have illustrated the characteristical behavior and the general dependencies on the system parameters of probability density $P_T(x_0, t)$ on figures (19), (20) and (21).

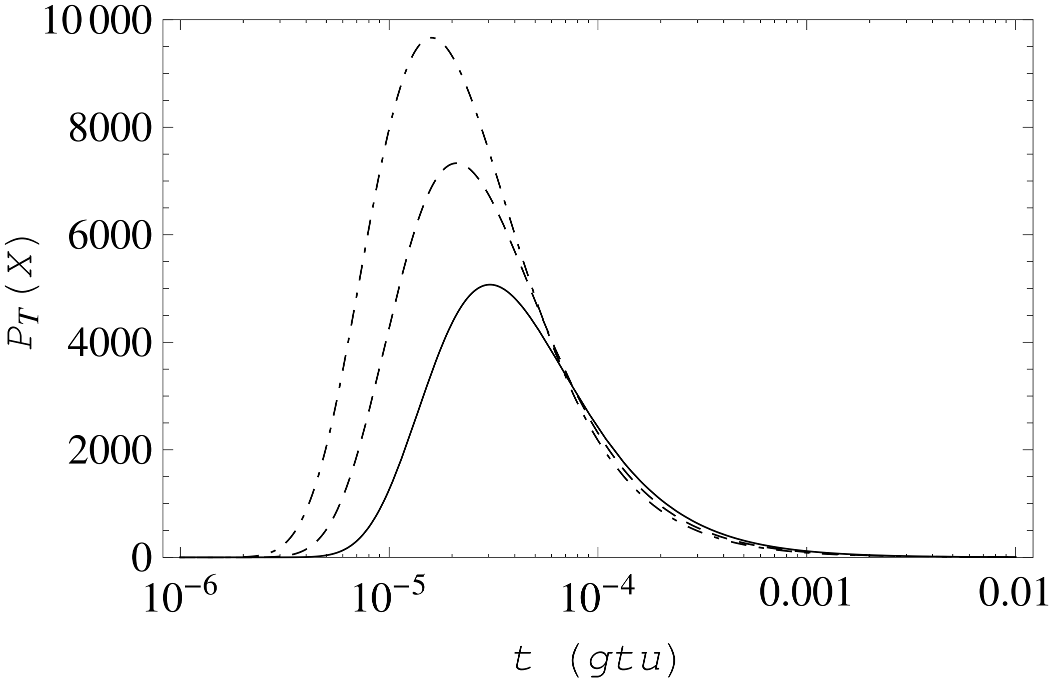

Figure 19: The probability density function of the lifetime of the population $P_T$ as a function of time $t$ at various values of the technological parameter $\alpha$ is illustrated. The parameter values: growth rate $r = 0.01\,(gtu)^{-1}$, noise intensity $D = 0.1\,(gtu)^{-1/2}$ and the initial population size $x_0 = 100$. Solid line: $\alpha_1 = 1.01$, with with corresponding mean lifetime of the population $\langle T_1 \rangle = 0.95\,(gtu)$ and the population's prospective lifetime $t_{p1} = 0.000030\,(gtu)$; dashed line: $\alpha_2 = 1.05$, with corresponding mean lifetime of the population $\langle T_2 \rangle = 0.76\,(gtu)$ and the population's prospective lifetime $t_{p2} = 0.000021\,(gtu)$; dashed-dotted line: $\alpha_3 = 1.08$, with corresponding mean lifetime of the population $\langle T_3 \rangle = 0.64\,(gtu)$ and the population's prospective lifetime $t_{p3} = 0.000016\,(gtu)$.

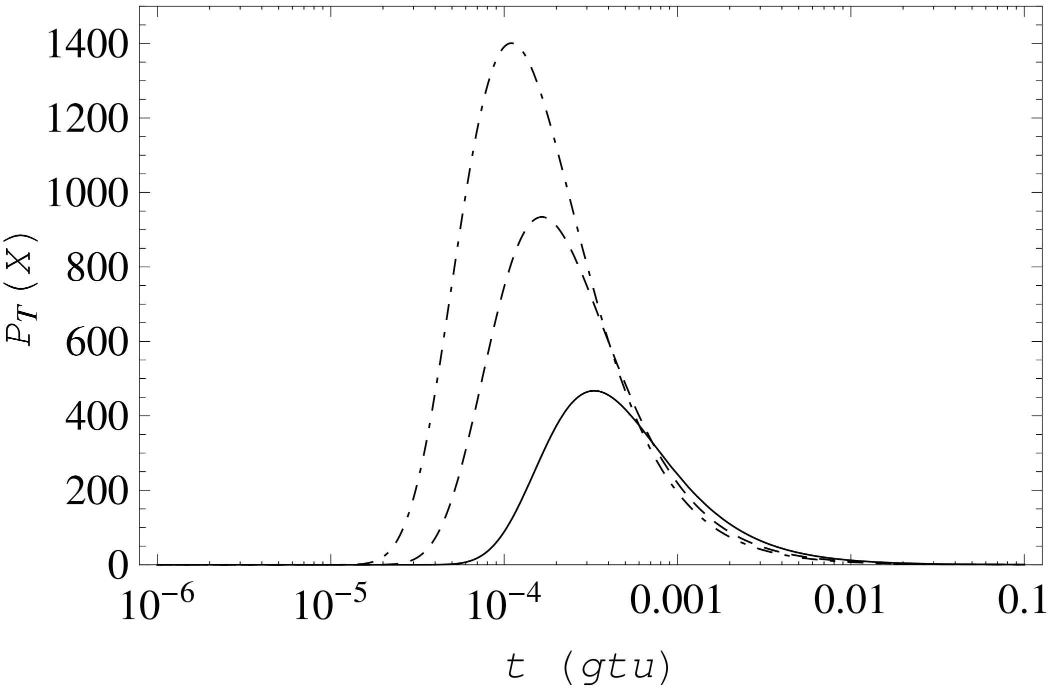

Figure 20: The probability density function of the lifetime of the population $P_T$ as a function of time $t$ at various values of the noise intensity $D$. The growth rate $r = 0.01\,(gtu)^{-1}$, the technological parameter $\alpha = 1.01$ and the initial population size $x_0 = 100$. Solid line: $D_1 = 0.1\,(gtu)^{-1/2}$ and the population's prospective lifetime $t_{p1} = 0.00033\,(gtu)$; dashed line: $D_2 = 0.2\,(gtu)^{-1/2}$, and the population's prospective lifetime $t_{p2} = 0.00017\,(gtu)$; dashed-dotted line: $D_3 = 0.3\,(gtu)^{-1/2}$ and the population's prospective lifetime $t_{p3} = 0.00011\,(gtu)$. The mean lifetime of the population for these parameters is $\langle T \rangle = 0.99\,(gtu)$.

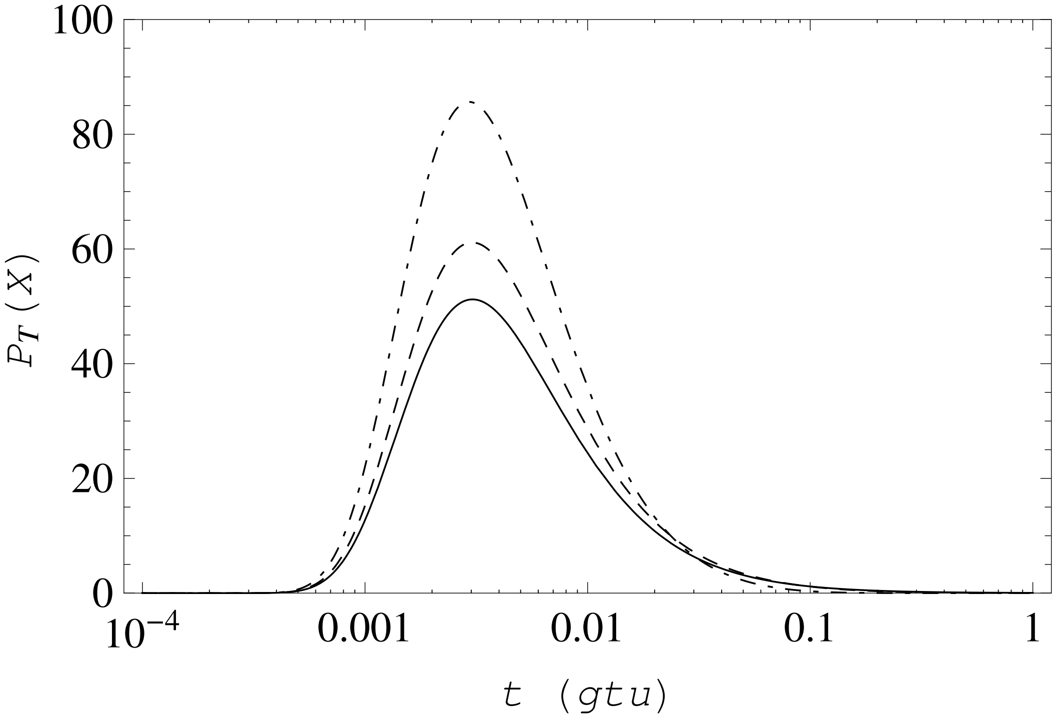

Figure 21: The probability density function of the lifetime of the population $P_T$ as a function of time $t$ at various values of the growth rate $r$ . The technological parameter $\alpha = 1.01$, the noise intensity $D = 0.01\,(gtu)^{-1/2}$ and the initial population size $x_0 = 100$. Solid line: $r_1 = 0.01\,(gtu)^{-1}$, with with corresponding mean lifetime of the population $\langle T_1 \rangle = 0.95\,(gtu)$ and the population's prospective lifetime $t_{p1} = 0.0030\,(gtu)$; dashed line: $r_2 = 0.2\,(gtu)^{-1}$, with with corresponding mean lifetime of the population $\langle T_2 \rangle = 0.047\,(gtu)$ and the population's prospective lifetime $t_{p2} = 0.0030\,(gtu)$; dashed-dotted line: $r_3 = 0.6\,(gtu)^{-1}$, with with corresponding mean lifetime of the population $\langle T_3 \rangle = 0.016\,(gtu)$ and the population's prospective lifetime $t_{p3} = 0.0029\,(gtu)$.

Figure (20) illustrates that with the increase of noise intensity, the probability density function $P_T(x_0, t)$ has a much higher maximum, however, the corresponding maximum point $t_p$ is much smaller. A similar behavior holds for the variation of technological parameter $\alpha$ and the growth rate $r$, plotted on figures (19) and (21).

The probability of encountering the demographic singularity around the population's prospective lifetime $t_p$ is larger for higher values of noise intensity $D$. Therefore, the quantity $t_p$ is a much better estimate for determining the time of demographic singularity for populations with strong inherent stochasticity. If the noise acting onto the growth rate is strong, then the estimate of the population's prospective lifetime $t_p$ is also more statistically relevant.

## 4.4. General Discussion and Applications to Other Systems

It should be reckoned that the model studied in this paper is applicable to a much broader class of systems. Explosive processes are common natural phenomena, often relating to systems exhibiting non-linear positive feedback. Such processes are relevant in modelling different biological systems, for instance, in ecological systems where two or more species are bound by facultative mutualism (Mao, Luo, 2007; Gora, 2005). Evolutionary processes, such as the growth of biodiversity in a biota, are also explosive by nature because of the accelerating returns between the diversity growth and community structure complexity (Markov, Korotayev, 2008). Subharmonic resonances in plasmas can also exhibit super-exponential growth (Mori, *et al.,* 2000). In addition, the present model can be applicable for modeling other types of critical phenomena such as, e.g. earthquakes (Tabar, *et al.,* 2006) and epileptic seizures (Mormann, et al., 2005). A relatively common application of this model is to predict stock market crashes. Similarly to the current work, Sornette and Andersen (2002) have put forward an analogous stochastic model to describe the super-exponential growth of financial bubbles. The macroscopic behaviour of humans can also be well described with this type of processes. Heterogeneity in their decision-making can often lead to critical phenomena at a macroscopic level (Sato, Takayasu, 1998; Saichev, Sornette, 2010).

Another key issue in modelling human population dynamics is how to estimate the environmental effects on the population. Since the work of Verhulst and Pearl on modelling environment as a kind of saturating effect acting upon a population, it has become clear that understanding the dynamics of the environment is crucial if we want to understand the behaviour of the dynamics of the population. It follows from the high level of complexity and the lack of detailed knowledge of the nature of the environment, that the latter should be analyzed via probabilistic models.

# CONCLUSION

In the present thesis we considered the super-exponential growth model with a stochastic growth rate. The historical estimates for the past human population growth indicate that human numbers have been growing faster than exponentially for most of the known history. The accelerating growth rate contains endogenously its own limit in the shape of finite-time singularity. Hence, even the optimistic view has to be revised. We proposed a model with a growth rate of immanent stochastic nature. Stochasticity arises from the environmental noise, which was modeled with a Stratonovich representation of Gaussian white noise. The heart of the research lies in an analytical description of the interplay between multiplicative white noise and the self-reinforcing behaviour of a system.

We observed from the model analysis that the mean value of the population size has a maximum, which tends to zero with the passage of time. This is due to the fact that we interpreted the singular state as an absorbing boundary. That means that the super-exponentially growing population is an irreversible process. Once, the population has reached the singular state, it cannot return to its previous behaviour.

The mean lifetime of the population $\langle T \rangle$ and the second moment of the lifetime $\langle T^2 \rangle$ have also been calculated. As well as the probability distribution of lifetime of the population $P_T$ and the population's prospective lifetime $t_p$. The population's mean lifetime coincided with the critical time observed in the deterministic model. However, since the CPDF of the mean lifetime of the population is asymmetric, the population's prospective lifetime differs significantly from the mean lifetime of the population. It transpired from the analysis that anywhere in the region $D \in (0, \infty)$, the ratio $0 < t_p / \langle T \rangle < 1$, which means that the population's prospective lifetime is always shorter than the mean lifetime of the system, when noise is introduced to the system. Moreover, the ratio $t_p / \langle T \rangle$ decays rapidly as a power law when $D \to \infty$. The same behaviour holds for others parameters as well, the ratio $t_p / \langle T \rangle$ decays rapidly as a power law when $\alpha \to \infty$ or $r \to \infty$, as long as, $D > 0$. We can conclude that if noise is introduced to the system the population's prospective lifetime is significantly smaller than the mean lifetime of the population $\langle T \rangle$.

We deduced from the results that the relative variance is much greater than 1 when $t > t_p$. If the relative variance is much greater than 1, it means that the CPDF is heavily asymmetrical and that strongly eccentric values of the population sizes in the population ensemble skew the CPDF. Therefore, the mean value is not a good estimate for the population's actual size and there is a great uncertainty about the population's actual size after $t > t_p$.

Author's opinion is that an interesting further development on this topic could be made by devising a stochastic model to describe the dynamics of the global human population carrying capacity.

# SUMMARY

In the study of human population biological, social, spatial, ecological, demographic, cultural and economic aspects are inextricably linked. Therefore, it is apparent that the holistic approach is needed to study the dynamics of human population dynamics. Another problem with modelling human population dynamics is the unpredictability, complexity and nonlinearity of the system. The focus of research in this thesis is to develop a relevant model to describe the dynamics of human population dynamics to estimate the future tendencies and the sustainability of the human population.

The aim of this thesis is to devise a mathematical model to describe the behaviour of super-exponentially growing population considering the inherent fluctuations within the demographics and perturbations of environmental origin, as well as uncertainties arising from incomplete information about the history of the dynamics. The noise has been assumed to act on the growth rate multiplicatively and is assumed to be Gaussian white noise a la Stratonovich. Based on wide range of meticulous empirical evidence on global and local scale, there are grounds to assume a strong correlation between the dynamics of population growth and the human technological innovations, as well as its economic output. Moreover, the super-exponential growth rate is a property of dynamical systems exhibiting endogenous nonlinear positive feedback, i.e. of self-reinforcing systems.

We calculated the exact solution for the CPDF $P(x,t|x_0,0)$, the mean value $\langle X(t) \rangle$, the relative variance $\delta(X)$ of the population size. The mean lifetime $\langle T \rangle$ and the second moment of the lifetime $\langle T^2 \rangle$ have also been calculated. As well as probability distribution of lifetime of the population $P_T$ and the population's prospective lifetime $t_p$. One of the most important results of the analysis of the stochastic super-exponential model was the derivation of the analytical formula for the population's prospective lifetime $t_p$. It followed from the analysis, that inequality $t_p < \langle T \rangle$ always holds if noise is introduced to the system. Therefore, the population's prospective lifetime $t_p$ of the stochastic model (in the presence of noise) is always shorter than the deterministic critical time $t_c$. We also observed that if $t > t_p$, then the mean value of the population size does not describe the actual population size very well, because the relative variance $\delta(X) >> 1$, when $t > t_p$. This is due to the fact that there are heavily eccentric values present and the CPDF of $X(t)$ is heavily skewed.

Our interpretations of the mathematical model have a special bearing to human population dynamics. The obtained results can also be applied to a variety of phenomena ranging from biology to plasma physics and to other types of processes that are explosive in their nature due to endogenous positive feedback.

In conclusion, introducing noise on the system did not change its behaviour qualitatively. However, quantitatively the behaviour did change. Most significantly, we showed that the population reaches the demographic singularity much sooner when noise is introduced to the system.

# KOKKUVÕTE

Käesoleva magistritöö „Analysis of the Stochastic Super-Exponential growth model" ("Stohhastilise supereksponentsiaalse kasvumudeli analüüs") raames on uuritud supereksponentsiaalselt ehk kiiremini kui eksponentsiaalselt kasvavat populatsiooni, mille kasvukiirus fluktueerub ajas. Demograafiliste andmete kohaselt on maailma rahvastik kasvanud supereksponentsiaalselt vähemalt viimase kahe tuhande aasta vältel. Selline inimpopulatsiooni arvukuse plahvatuslik kasv on olnud võimalik tänu tehnoloogilise progressi saavutustele, millele on omakorda toetunud supereksponentsiaalselt kasvav majandus.

Seetõttu tuleks rahvastiku kasvu uurimisel arvestada ka ühiskonna tehnoloogilist innovatiivsusvõimekust, sest populatsiooni arvukuse ja tehnoloogilise progressi vahel toimub positiivne tagasiside. Inimeste arvukuse kasv toob endaga kaasa uute ideede ja teadmiste kasvu, mis võimaldab inimkonnal suurendada oma keskkonnamahutavust ning keskkonnamahutavuse suurenemine võimaldab omakorda populatsioonil kiiremini kasvada. Selline süsteemi iseregulatsioon on võimaldanud inimpopulatsioonil kasvada kiiremini kui eksponentsiaalselt. Sellele viitavad nii andmed globaalsel tasemel (Johansen, Sornette, 2001) kui ka uuringud rahvastiku ja tehnoloogilise progressi vallas erinevates linnades (Bettencourt et al., 2007). Samuti on Myrskylä, et al (2009) näidanud, et sündivuse ja sotsioloogilise ning majandusliku arengutaseme vahel on tugev positiivne korrelatsioon juhul kui viimane ületab teatavat läveväärtust. Demograafilise ülemineku teooria ei ole lõplikult suutnud seletada senini toimunud demograafilisi nähtusi (Cohen, 1995). Seega on veel avatud küsimus, kas me saame oodata demograafilise ülemineku toimumist tänapäeva Aasia, Aafrika ja Ladina-Ameerika vähemarenenud riikides.

Supereksponentsiaalselt kasvavat populatsiooni kirjeldav matemaatiline mudel omab lõpliku aja jooksul ilmnevat singulaarsust, st inimeste arv kasvab lõpliku aja vältel lõpmatusse. Seda matemaatilist idealisatsiooni võib tõlgendada kui populatsiooni jõudmist katastroofi, mis võib tähendada populatsiooni väljasuremist selle vahetus tähenduses. Samuti, tuues paralleeli mittetasakaaluliste protsesside vallast, võib demograafilist singulaarsust tõlgendada ka kui rahvastiku kasvudünaamika "faasiülemikut" uut sorti režiimile. Võime samuti eeldada, et seda sorti järsk üleminek on tugevalt pöördumatu protsess.

Supereksponentsiaalse mudeli alusel saab populatsiooni jaoks arvutada välja katastroofi jõudmise aja. Vastavat ajamomenti nimetatakse kriitiliseks ajaks ning Johanseni ja Sornette' (2001) hinnangul asub inimpopulatsiooni jaoks kriitiline ajamoment vahemikus 2052±10. Viimati mainitud töös kasutati mitteparameetrilist testi tuvastamaks logaritm-perioodilist käitumist, et määrata kriitilist ajamomenti. Otsene lähendamine mudelile on raskendatud, sest andmestikus esineb tugev müra.

Käesoleva magistritöö uurimuslikus osas on esitatud ja analüüsitud supereksponentsiaalset kasvumudelit, mille puhul on kasvuparameetrit käsitletud juhuslikult fluktueeruva suurusena, mis käitub analoogselt Gaussi valge müra protsessiga a la Stratonovich. Stohhastilise mudeli koostamise eesmärgiks on uurida müra rolli ning mõju populatsiooni dünaamikas.

Antud kontekstis võime müra all mõelda demograafilisi ja keskkonnast pärinevaid fluktuatsioone ning samuti ka ebatäielikust andmestikust tulenevat määramatust. Vastav käsitlus rajaneb eeldusel, et inimpopulatsiooni terviklikku kasvukiirust mõjutavad korraga väga palju erinevaid protsesse ning ükski kasvukiirust mõjutav tegur ei ole eelistatud seisundis. Ühtlasi lähtume eeldusest, et inimpopulatsiooni dünaamilise süsteemi analüüs toimub väga laias ajavahemikus. Arvestame, et demograafiliste mehhanismide ajaskaala on palju suurem kui kasvuparameetrile avalduvate fluktuatsioonide ajaskaala.

Leidsime, et multiplikatiivse valge müra lisamine ei toonud endaga kaasa süsteemi käitumise kvalitatiivset muutust - stohhastilises populatsioonimudelis esineb samuti demograafiline singulaarsus. Koostatud stohhastilise supereksponentsiaalse kasvuga mudeli analüüsi osas arvutasime populatsiooni arvukuse tõenäosustiheduse, keskväärtuse, suhtelise dispersiooni. Mudeli analüüsi käigus ilmnes, et populatsiooni arvukuse keskväärtus omab maksimumi, mille väärtus väheneb aja möödudes. Suhtelise dispersiooni väärtused tõusid populatsiooni ajalise evolutsioneerumise käigus kiiresti üle ühe, mis viitab tõenäosusjaotuse ebasümmeetrilisusele. Uurisime ka kuidas erinevad süsteemi parameetrid nagu kasvuparameeter, tehnoloogilise progressi parameeter ja müra intensiivsus mõjutavad populatsiooni arvukuse keskväärtuse ja suhtelise dispersiooni ajalist käitumist.

Avaldasime populatsiooni eluea tõenäosusjaotuse. Populatsiooni eluea all mõistame antud kontekstis populatsiooni demograafilisse singulaarsusesse jõudmise aega. Samuti arvutati välja populatsiooni keskmise eluea ja tõenäoseima eluea eksaktsed valemid. Leidsime, et populatsiooni keskmise eluea väärtus langeb kokku deterministliku supereksponentsiaalse mudeli demograafilise singulaarsuse kriitilise ajamomendiga. Tõenäoseim eluiga on alati väiksem populatsiooni keskmisest elueast, kui süsteemis esineb müra. Järelikult, kui populatsiooni kasvukiirust käsitleda fluktueeruva suurusena on tõenäoline, et demograafilisse singulaarsusesse jõudmine toimub isegi varem kui deterministlik mudel seda ennustab.

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